Chapter 1

Heat Propagation in 3D Solids

In order to obtain the temperature distribution $T(x, y, z, t)$ in a 3D body as a function of the point $P(x, y, z)$ and of time $t$, it is necessary to study the problem of heat conduction, one of the three kinds of heat transfer. The remaining two, that is convection and radiation have only a limited, although very significant, role in the determination of temperature distribution in the case solids and structures are concerned. Heat conduction is based on two basic principles of classical physics:

- heat is transferred from high temperature points to low temperature points of the solid, depending upon a material properties called thermal conductivity;
- the accumulation of heat in a material point of the body increases the temperature of
that point, depending upon another property of the material called specific heat. In order to set the mathematical model for the heat conduction problem we assume that a 3D material body is occupying a volume $V$ with boundary $\partial V$ in a 3D cartesian framework $(O, x, y, z)$.

For each point $P(x, y, z)$ we define the temperature $T(x, y, z, t)$ as a continuous function of space and time coordinates and, considering an isothermal surface of normal $n$ passing through $P$, we define the heat flux $q_n$ as the heat $Q$ per unit surface and per unit time passing through the surface. The temperature and the heat flux are usually measured, respectively, in $K$ and in $W/m^2$. The heat flux has the physical dimension of

$$[q] = [F \cdot L \cdot L^{-2} \cdot T^{-1}].$$

By considering the derivative of $T$ along $n$, and remembering that the heat flux $q_n$ is directed from high temperatures to low temperatures, that is in the direction opposite to the normal $n$, the first principle of heat conduction can be written as

$$q_n = -\lambda_n \frac{\partial T}{\partial n}$$

(1.1)

where $\lambda_n$ is the thermal conductivity of the solid along $n$, measured in $W/m \cdot K$. In an anisotropic body the thermal conductivity is different in all directions. It is possible to demonstrate that once the heat flux is known along three orthogonal directions, namely the coordinates $x, y, z$, one can obtain the value of $q_n$ provided that direction of $n$ is given. By

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**Figure 1.2:** The isothermal surface element $dS$ of normal $n$ passing through the point $P$
assuming that $\lambda_x, \lambda_y, \lambda_z$ are the thermal conductivities measured along $x, y$ and $z$, the three components of heat flux $q$ can be written as

$$ q_x = -\lambda_x \frac{\partial T}{\partial x}; \quad q_y = -\lambda_y \frac{\partial T}{\partial y}; \quad q_z = -\lambda_z \frac{\partial T}{\partial z}. \quad (1.2) $$

We now consider the second principle based on which the governing equations of heat transfer are obtained. At point $P$ we consider the balance of thermal energy of an infinitesimal volume of the solid. For sake of simplicity we consider a volume element $dxdydz$. We consider the amount of heat entering the element through all its surfaces. Through the surface $dydz$ passing through $P$ the portion of heat enters the volume in the $x$ direction in the time interval $dt$ is

$$ (dQ_x)_{in} = q_x dydzdt = -\lambda_x \frac{\partial T}{\partial x} dydzdt. \quad (1.3) $$

At the surface distant $dx$ from the previous one, the value of $q_x$ will be varied due to the fact that we move from the coordinate $x$ to the coordinate $x + dx$, and the heat is going out of the material portion

$$ (dQ_x)_{out} = \left( q_x + \frac{\partial q_x}{\partial x} \right) dydzdt = - \left[ \lambda_x \frac{\partial T}{\partial x} + \frac{\partial}{\partial x} \left( \lambda_x \frac{\partial T}{\partial x} \right) dx \right] dydzdt. \quad (1.4) $$

The amount of heat remaining in the volume due to the heat flow along the $x$ direction is

$$ (dQ_x)_{in} - (dQ_x)_{out} = \frac{\partial}{\partial x} \left( \lambda_x \frac{\partial T}{\partial x} \right) dx dydzdt. \quad (1.5) $$
Similar expressions can be obtained for the flow in y and z directions

\[(dQ_y)_\text{in} - (dQ_y)_\text{out} = \frac{\partial}{\partial y} \left( \lambda_y \frac{\partial T}{\partial y} \right) dx dy dz dt,\]  
(1.6)

and

\[(dQ_z)_\text{in} - (dQ_z)_\text{out} = \frac{\partial}{\partial z} \left( \lambda_z \frac{\partial T}{\partial z} \right) dx dy dz dt.\]  
(1.7)

The amount of heat that remains in the volume \(dx dy dz\) due to heat conduction through the surface of the elemental volume in the time interval \(dt\) is

\[dQ_1 = - \left( \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} \right) dx dy dz dt = \text{div}(\mathbf{q}) dx dy dz dt = \left[ \frac{\partial}{\partial x} \left( \lambda_x \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left( \lambda_y \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left( \lambda_z \frac{\partial T}{\partial z} \right) \right].\]  
(1.8)

In presence of an interval heat generation per unit volume per unit time \(Q\), the amount of heat generation in the volume \(dx dy dz\) in the time interval \(dt\) is

\[dQ_2 = Q dx dy dz dt.\]  
(1.9)

On the other side the heat \(dQ_3 = dQ_1 + dQ_2\) will produce an increase of the temperature \(dT\) that is related to \(dQ_3\) as follows

\[dQ_3 = c \rho dT dx dy dz,\]  
(1.10)

where \(c\) is the specific heat for unit mass, measured in J/kg·K, and \(\rho\) is the density, measured in kg/m³. Since \(dQ_3 = dQ_1 + dQ_2\), by dividing both members by \(dx dy dz dt\) we obtain

\[c \rho \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left( \lambda_x \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left( \lambda_y \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left( \lambda_z \frac{\partial T}{\partial z} \right) + Q.\]  
(1.11)

This equation, called Fourier equation or heat conduction equation, holds for a nonhomogeneous anisotropic solid. It is partial differential equation of the second order in space variables and of the first order in time. For an homogeneous body \(\lambda_x, \lambda_y\) and \(\lambda_z\) do not depend upon the point \(P(x, y, z)\) and the equation reduces to

\[c \rho \frac{\partial T}{\partial t} = \lambda_x \frac{\partial^2 T}{\partial x^2} + \lambda_y \frac{\partial^2 T}{\partial y^2} + \lambda_z \frac{\partial^2 T}{\partial z^2} + Q.\]  
(1.12)

If we add the simplifying of isotropic body, for which

\[\lambda_x = \lambda_y = \lambda_z = \lambda\]  
(1.13)

we obtain

\[c \rho \frac{\partial T}{\partial t} = \lambda \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + Q.\]  
(1.14)
that can also be written as
\[
c_p \frac{\partial T}{\partial t} = \lambda \nabla^2 T + Q, \tag{1.15}
\]
where \( \nabla^2 (\cdot) = \frac{\partial^2 (\cdot)}{\partial x^2} + \frac{\partial^2 (\cdot)}{\partial y^2} + \frac{\partial^2 (\cdot)}{\partial z^2} \) is the \textit{laplacian operator}. Introducing the so called \textit{thermal diffusivity}
\[
k = \frac{\lambda}{c_p} \tag{1.16}
\]
measured in \( m^2/s \), equation 1.15 can be recast in the form
\[
\frac{1}{k} \frac{\partial T}{\partial t} = \lambda \nabla^2 T + \frac{Q}{\lambda}. \tag{1.17}
\]
If no heat is generated inside the volume the equation reduces to
\[
\frac{1}{k} \frac{\partial T}{\partial t} = \lambda \nabla^2 T \tag{1.18}
\]
If no variation in time is observed (stationary case) we have
\[
\lambda \nabla^2 T + \frac{Q}{\lambda} = 0, \tag{1.19}
\]
and if \( Q = 0 \) the equation reduces to the \textit{Laplace equation}:
\[
\lambda \nabla^2 T = 0. \tag{1.20}
\]

In order to complete the mathematical model for the heat conduction problem, initial and boundary condition should be set up. For initial condition, considering that the Fourier equation is first order in time, it is necessary to establish the value for temperature for any point \( P(x, y, z) \) of the volume \( V \) at the initial time \( t_0 \)
\[
T(x, y, z, t_0) = \overline{T}(x, y, z). \tag{1.21}
\]

As far as boundary equations, since the Fourier equation is second order in space variables, it is possible to assign at the boundary \( \partial V \) of the volume \( V \) occupied by the body either the temperatures or the derivatives in space of the temperatures, that is the heat fluxes. Assigned temperature on the boundary \( \partial V \)
\[
T(x, y, z, t) = \overline{T}(x, y, z) \quad \text{for} \quad P(x, y, z) \in \partial V. \tag{1.22}
\]
Assigned heat flux on the boundary \( \partial V \), being \( n \) the normal to the boundary at the point \( P \)
\[
q_n(x, y, z, t) = \overline{q}_n(x, y, z, t) \quad \text{for} \quad P(x, y, z) \in \partial V, \tag{1.23}
\]
that can also be written as
\[-\lambda \frac{\partial T(x, y, z, t)}{\partial n} = \eta_n(x, y, z, t). \quad (1.24)\]

In case of contact between the boundary of the solid and a fluid at temperature \( \theta \), the heat transfer can occur by convection. In this case
\[\eta_{nc} = h_c (T(x, y, z, t) - \theta), \quad (1.25)\]
where \( h_c \) is the convection heat transfer coefficient, measured in \( \frac{W}{m^2K} \).

A body at temperature \( T \) loses heat by radiation if surrounded by a medium at temperature \( \theta \). The heat flux is in this case
\[\eta_{nr} = \sigma F \left( T^4(x, y, z, t) - \theta^4 \right), \quad (1.26)\]
being \( \sigma = 5.67 \times 10^{-8} W/m^2 \cdot K^4 \) the Stefan-Boltzmann constant and \( F \) the emissivity of the surface. For a not too large difference between \( T \) and \( \theta \)
\[\eta_{nr} = \sigma F \left( T^2 - \theta^2 \right) \left( T^2 + \theta^2 \right) = \sigma F (T - \theta) (T + \theta) (T^2 + \theta^2) \]
\[\simeq \sigma F (T - \theta) 2\theta^2 \theta^2 = 4\sigma F \theta^3 (T - \theta). \quad (1.27)\]
And, if we assume \( h_2 = 4\sigma F \theta^3 \) we can write
\[\eta_{nr} = h_2 (T - \theta) \quad (1.28)\]
in a form similar to the one adopted for the convection. In the case that two bodies 1 and 2 are in contact the following conditions hold on the points belonging to the contact surfaces of normal \( n \)
\[T_1 = T_2 \quad (1.29)\]
\[\lambda_1 \frac{\partial T_1}{\partial n} = \lambda_2 \frac{\partial T_2}{\partial n}. \quad (1.30)\]