Consider a conic nozzle feeded by a tank containing air (R=287 J/Kg K, $\gamma=1.4$), with a total temperature $T_0=2000\,K$. The maximum mass flow rate through the nozzle is $Q_{max}=1\,Kg/s$. Under design conditions the pressure at the nozzle exit is $p_e=10^4\,Pa$ and the Mach number $M_e=3$.

1) Determine the area of the exit section A_e and the area of the throat A_T.

2) Find the values of the external pressure that determine different working regimes of the nozzle

Solution:

1) Using the fact that M_e = 3 we can use the table for isentropic flow tables to obtain,
$A_e/A_T=4.2346$.
Further using the table we can determine the pressure inside the tank and the temperature at the exit, under design conditions,

$$p_e/p_0=0.027244 \rightarrow p_0=367323\,Pa$$
$$T_e/T_0=0.35714 \rightarrow T_e=714.28\,K\,.$$

The speed of sound and the density at the nozzle exit can also be computed,

$$a_e=\sqrt{(\gamma R T_e)}=535.72\,m/s$$
$$\rho_e=p_e/(RT_e)=0.04878\,Kg/m^3$$

Using the mass flow rate we can evaluate the exit area,

$$A_e=Q_{max}/(\rho_e M_e a_e)=0.01276\,m^2$$

and the throat are

$$A_T=0.003\,m^2 \quad .$$

2) The flow in a converging-diverging nozzle is characterize by three critical pressures, $p_{e1}\,p_{e2}\,p_{e3}$ which identify different working regimes.
The first pressure p_{e1} can be computed making the hypothesis that the flow in the nozzle is subsonic. Under this conditions, from the isentropic flow tables we find,
$$A_e/A_T=4.2346 \rightarrow M_e=0.13834$$
$$p_{e1}/p\,01=0.98672 \rightarrow p_{e1}=362445\,Pa$$
The pressure p_{e3} can be computed making the hypothesis that the flow in the nozzle is supersonic and the flow isentropic,
$$A_e/A_T=4.2346 \rightarrow M_e=3$$
$$p_{e3}/p_{01}=0.027244 \rightarrow p_{e3}=10^4\,Pa$$
Finally pressure p_{e2} can be computed making the hypothesis that the flow is supersonic in the divergent and that a normal shock is present at the exit section. Since the flow is isentropic the Mach number upstream of the shock is M=3. Using the normal shock jump relations we find the Mach number dowstream of the shock and the ratio of the total pressures across the shock,

$$M_1=3 \rightarrow M_2=0.47519$$
$$p_{02}/p_{01}=0.32834$$
and from the isentropic flow tables,
$$M=0.4719 \rightarrow p_{e2}/p_{02}=0.85676 \rightarrow p_{e2}=p_{e2}/p_{02}*p_{02}/p_{01}*p_{01}=103331\,Pa$$

Consider a converging-diverging nozzle with throat area $A_T = 10\,cm^2$. The ratio between the exit and the throat area is $A_e / A_T = 2$. The nozzle is connected to a tank filled with a gas with R=297J/KgK and , total temperature $T_0 = 1000\,K$ and total pressure $p_0 = 10^5\,Pa$.
Determine:

1) The external pressure p_d such that the nozzle operates under design conditions.
2)The external pressure for which we have a normal shock at the exit section.
3)The external pressure for which there is a normal shock at $A_s / A_T = 1.5$.
4)The mass flow rate when the external pressure is $p_s = 0.96\,p_0$.
5) The maximum mass flow rate

Solution

1) We can immediately determine the exit area and the density inside the tank,

$\rho_0 = p_0 / (T_{0R}) = 0.337$

$A_e = 2 A_0 = 0.002 m^2$

Using the isentropic gas table, we can determine the exit conditions from the area ratio

$A_e / A_T = 2 \rightarrow$

$M_e = 2.20$

$p_e = 9352.2 Pa$

$T_e = 508.13 K$

$\rho_e = 0.062 Kg/m^3$

$a_e = \sqrt{\gamma R T_e} = 459.65 m/s$

The nozzle operates under design conditions when the pressure at the exit is equal to the external pressure, so $p_p = p_e$.

2) The flow is isentropic up to the exit section, so that the Mach number upstream of the shock is

$M_1 = 2.2 \rightarrow M_2 = 0.54706$ and $p_{02} / p_{01} = 0.62814$

From the isentropic flow table we find $M_2 = 0.55 \rightarrow p_2 / p_{02} = 0.81417$ and the external pressure

$p_e = p_2 = p_2 / p_{02} * p_{02} / p_{01} * p_{01} = 51141 Pa$

3)From the isentropic flow tables,

$A_s / A_T = 1.5 \rightarrow M_1 = 1.86$, $p_1 / p_{01} = 0.15873$ and from the normal shock tables $M_2 = 0.62363$, $p_2 / p_1 = 3.8695$. We can find $p_2 = p_1 * p_2 / p_1 = 61420.6 Pa$.

Again, from the isentropic flow tables we find $A_2 / A_2^* = 1.1882$, $p_2 / p_{02} = 0.784$ and

$A_e / A_2^* = A_e / A_2 * A_2 / A_2^* = 1.7823$. Using the isentropic flow tables with this aspect ratio we find $M_e = 0.35$ and $p_e = p_e / p_{02} * p_{02} / p_2 * p_2 = 71979 Pa$.

4)If the external pressure is $p_s = 0.96 p_0$, from the isentropic tables we find $M_s = 0.24$ and,

$\rho_s = 0.327 Kg/m^3$

$T_s = 988.61 K$

$a_s = 641 m/s$

$Q_s = \rho_s u_s A_e = \rho_s a_s M_s A_e = 0.101 Kg/s$

5)The maximum mass flow rate is reached when the throat is in choking so, using the value from point 1)

$Q_{max} = \rho_e M_e a_e A_e = 0.1259$