PROBLEM 3

A convergent-divergent nozzle is designed to expand air from a chamber in which the pressure is 800 kPa and the temperature is 40°C to give a Mach number of 2.5. The throat area of the nozzle is 0.0025 m^2. Find (1) the flow rate through the nozzle under design conditions, (2) the exit area of the nozzle, (3) the design back pressure and the temperature of the air leaving the nozzle with this back pressure, (4) the lowest back pressure for which there is only subsonic flow in the nozzle, (5) the back pressure at which there is a normal shock wave on the exit plane of the nozzle, (6) the back pressure below which there are no shock waves in the nozzle, (7) the range of back pressures over which there are oblique shock waves in the exhaust from the nozzle, (8) the range of back pressures over which there are expansion waves in the exhaust from the nozzle, (9) the back pressure at which a normal shock wave occurs in the divergent section of the nozzle at a point where the nozzle area is half way between the throat and the exit plane areas.

SOLUTION

It will be assumed that the supply chamber is large enough to be able to assume that the chamber pressure is the stagnation pressure and that the chamber temperature is the stagnation temperature i.e.:

$$p_0 = 800 \text{ kPa and } T_0 = 313 \text{ K}$$

Part (1)

When the nozzle is operating at the design conditions the Mach number will be one at the throat. The mass flow rate through the nozzle will therefore be given by:

$$\dot{m} = \rho^* V^* A^* = \rho^* a^* A^*$$

For a Mach number of 1 the software for isentropic flow gives:

$$\frac{T_0}{T^*} = 1.200 \ , \ \frac{p_0}{p^*} = 1.893$$

Hence:

$$T^* = \frac{T_0}{T_0 / T^*} = \frac{313}{1.200} = 260.8 \text{ K}$$

and:

$$p^* = \frac{p_0}{p_0 / p^*} = \frac{800}{1.893} = 422.6 \text{ kPa}$$

Therefore:

$$\rho^* = \frac{p^*}{R\,T^*} = \frac{422600}{287 \times 260.8} = 5.646 \text{ kg / m}^3$$

and:

$$a^* = \sqrt{\gamma\, R\, T^*} = \sqrt{1.4 \times 287 \times 260.8} = 323.7 \text{ m/s}$$

and:

$$A^* = 0.0025 \text{ m}^2$$

Hence:

$$\dot{m} = \rho^*\, a^*\, A^* = 5.646 \times 323.7 \times 0.0025 = 4.659 \text{ kg/s}$$

Therefore the mass flow rate through the nozzle is therefore 4.659 kg/s.

Part (2)

The nozzle is designed to generate an exit Mach number of 2.5. For this Mach number the software for isentropic flow gives:

$$\frac{A_e}{A^*} = 2.637\ , \quad \frac{T_0}{T_e} = 2.250\ , \quad \frac{p_0}{p_e} = 17.09$$

A_e being the nozzle exit area and T_e and p_e being the temperature and pressure on the nozzle exit plane respectively.

Hence:

$$A_e = \frac{A_e}{A^*}\, A^* = 2.637 \times 0.0025 = 0.0066 \text{ m}^2$$

Therefore the exit plane area of the nozzle is 0.0066 m².

Part (3)

When the nozzle is operating at this design condition, i.e. when the nozzle is perfectly expanded, there are no shock waves in the flow and $p_e = p_b$ and as noted above:

$$\frac{T_0}{T_e} = 2.250 \ , \quad \frac{p_0}{p_e} = 17.09$$

Therefore, when the nozzle is operating at the design condition:

$$p_e = \frac{p_0}{p_0 / p_e} = \frac{800}{17.09} = 46.81 \ \text{kPa}$$

and:

$$T_e = \frac{T_0}{T_0 / T_e} = \frac{313}{2.250} = 139.1 \ \text{K}$$

Therefore, the design back pressure is 46.81 kPa and the temperature of the air leaving the nozzle at the design conditions is 139.1 K (= -133.9°C).

Part (4)

The highest back pressure, p_b , that will give choking at the throat is that which gives a Mach number of one at the throat and which involves subsonic flow in the divergent section of the nozzle with the result that $p_e = p_b$, p_e being the pressure on the nozzle exit plane. For this case then the subsonic isentropic flow software gives for $A_e / A^* = 2.637$:

$$\frac{p_0}{p_e} = 1.036$$

Hence in this case:

$$p_e = \frac{p_0}{p_0 / p_e} = \frac{800}{1.036} = 772.2 \ \text{kPa}$$

Therefore the lowest back pressure at which there is only subsonic flow in the nozzle is 772.2 kPa.

Part (5)

When there is a normal shock wave at the exit, the flow in the nozzle is shock free with a supersonic velocity at the exit. Hence, if there is a shock wave at the exit, the Mach number ahead of the shock wave will be 2.5 and p_e = 46.81 kPa. For a Mach number of 2.5, the software for normal shock waves gives the pressure ratio across the wave as:

$$\frac{p_b}{p_e} = 7.125$$

Here it has been noted that with a normal shock wave at the exit the flow downstream of the shock wave will be subsonic and the pressure downstream of the shock wave must therefore be equal to the back pressure, p_b . Hence, when there is a normal shock wave at the exit:

$$p_b = \frac{p_b}{p_e} p_e = 7.125 \times 46.81 = 333.5 \text{ kPa}$$

Therefore there is a normal shock wave on the nozzle exit plane when the back pressure is 333.5 kPa.

Part (6)

There are no shock waves in the nozzle for back pressures below that required to give a normal shock on the exit plane. Hence, there are no shock waves in the nozzle for back pressures below 333.5 kPa.

Part (7)

Oblique shock waves will occur in the exhaust from the nozzle for back pressures that are below that which will give a normal shock wave on the exit plane and above the design back pressure i.e for back pressures that are between 333.5 kPa and 46.81 kPa.

Part (8)

Expansion waves will occur in the exhaust from the nozzle for back pressures that are below the design back pressure i.e for back pressures that are less than 46.81 kPa.

Part (9)

Because the nozzle has an area ratio of:

$$\frac{A_e}{A^*} = 2.637$$

the normal shock wave therefore occurs at a point in the divergent section of the nozzle at which:

$$\frac{A}{A^*} = \frac{0.5\,(A_e + A^*)}{A^*} = 0.5\,\frac{A_e}{A^*} + 0.5 = 0.5 \times 2.637 + 0.5 = 1.819$$

Isentropic software gives, using this area ratio, the Mach number and pressure ratio just upstream of the shock wave as:

$$M_s = 2.089\,, \quad \frac{p_0}{p_s} = 8.992$$

the subscript s denoting conditions just upstream of the shock wave.

Hence, the pressure just upstream of the shock wave is given by:

$$p_s = \frac{p_0}{p_0 / p_s} = \frac{800}{8.992} = 88.97 \text{ kPa}$$

Next consider the flow across the shock wave. Software for normal shock waves for a Mach number of 2.089 gives:

$$M_d = 0.5629\,, \quad \frac{p_d}{p_s} = 4.925$$

the subscript d denoting conditions just downstream of the shock wave.

Hence, the pressure just downstream of the shock wave is given by:

$$p_d = p_s\,\frac{p_d}{p_s} = 88.97 \times 4.925 = 438.2 \text{ kPa}$$

For the flow just downstream of the shock wave where the Mach number is 0.5629 isentropic flow software gives:

$$\frac{A_d}{A_d^*} = 1.236\,, \quad \frac{p_{0d}}{p_d} = 1.240$$

Using this value gives:

$$\frac{A_e}{A_d^*} = \frac{A_e}{A_d}\,\frac{A_d}{A_d^*} = \frac{2.637}{1.819} \times 1.236 = 1.792$$

For this area ratio the software for subsonic isentropic flow gives:

$$\frac{p_{0d}}{p_e} = 1.087$$

Therefore:

$$p_e = \frac{p_{0d} / p_d}{p_{0d} / p_e} p_d = \frac{1.240}{1.087} \times 438.2 = 499.9 \text{ kPa}$$

Because the flow downstream of the shock wave is subsonic this will be equal to the back pressure.

Therefore the normal shock wave will occur at this point when the back pressure is 499.9 kPa.