Chapter 7

Dynamics of Space-Continuous Structures
- Eigenfunctions method

This Chapter is dedicated to the analytical methods to represent the time and space domain solutions of linearized Partial Differential Equations, PDE (free response and driven cases), using eigenfunctions associated to the structural operator of the system or equivalent concepts as the influence function method (Sec. 7.1); some issue on the truncated forms of this serie solutions are also discussed in Sec. 7.2. These types of PDEs typically describe the dynamic behavior of simplified structure models as bars, beams, plates, shells, etc.; nevertheless, the shown procedures are quite general and in Sec. 7.3 it is shown how they can be applied to a generic 3D linear elastic static problem.

7.1 Eigenfunctions Method – Linear Differential Operators Theory

In the present Section, after some preliminary concepts concerning the definition of Linear differential operator (Sec. 7.1.1), eigenvalues and eigenvectors of a linear operator (Sec. 7.1.2), inner product operation (Sec. 7.1.3) and self-adjointness (Secs. 7.1.4 and 7.1.6), the eigenfunction method is presented as a procedure to solve PDE problems of statics and dynamics of structures (Sec. 7.1.7); i.e., PDE problems characterized by a linear operator (structural operator) which is intrinsically self-adjoint.
7.1.1 Definition of Linear Differential Operator

Let us consider (the scalar) differential operator $L$ which operates on the space of the (scalar) functions $f(x), g(x)$, that are generally functions of $N$ independent variables $\mathbf{x} = \{x_1, x_2, ..., x_N\}^T$. Indeed, for the applications considered in the following, $N = 3$ for three-dimensional space problems. For the sake of simplicity, we shall assume that domain of definition of such functions is a closed and limited domain. The differential operator $L$ is linear if (and only if)

$$L[\alpha f(x) + \beta g(x)] = \alpha L[f(x)] + \beta L[g(x)]$$

(7.1)

for all values of $\alpha$ and $\beta$ which are defined in the field of real numbers.\(^1\) Some simple examples of linear operators are given by the ordinary derivatives of functions of one real variable, or by differential operators like the divergence, the gradient, the laplacian, the curl.

It has to be pointed out that the definition of linear differential operator includes the boundary condition to be satisfied by the generic functions $f(x)$ and $g(x)$; this issues will be clarify later introducing the concept of self-adjoint operator. Some examples of structural linear operators

$$
\begin{align*}
EIw^{IV} + \mu \ddot{w} &= f \\
-LAu^{II} + \mu \ddot{u} &= h \\
D
\nabla^4 w + \rho h \ddot{w} &= g \\
-\frac{\partial}{\partial x_j} \left( C_{ijkm} \frac{\partial u_k}{\partial x_m} \right) + \rho \ddot{u}_i &= \rho f_i \\
L_{ij} \dot{u}_j + \rho \ddot{u}_i &= \rho f_i
\end{align*}
$$

(bending beam)

(axial beam)

(bending plate)

(3D elastic problem)

Note that in the last example, $L_{ij}$ is a $3 \times 3$ matrix of space-differential operator.

7.1.2 Eigenvalues and Eigenfunctions of a Linear Operator

Let us consider a linear operator $L$ together with its boundary conditions: the eigenvalues $\lambda_n$ and the eigenfunctions $\phi_n(\mathbf{x})$ are defined respectively as the set of constants and the set of functions satisfying the differential problem

$$L\phi_n(\mathbf{x}) = \lambda_n \phi_n(\mathbf{x}) + B.C.$$ 

(7.2)

where with $B.C.$ have been indicated the associated boundary conditions.

\(^1\)Note also that the above definition includes the possibility to consider an arbitrary number of terms for the sum.
7.1.2.1 Example of eigenvalues and eigenfunctions

Let us consider the operator

$$L = \frac{d^2}{dx^2} + k \quad u = 0 \quad \text{for } x = 0 \text{ and } x = l$$ (7.3)

Thus the Eigenproblem is

$$\frac{d^2\phi_n}{dx^2} - (\lambda_n - k)\phi_n = 0$$ (7.4)

Let (for convience) $$-\omega_n^2 = \lambda_n - k$$ then

$$\frac{d^2\phi_n}{dx^2} + \omega_n^2\phi_n = 0$$ (7.5)

The characteristic equation is

$$\alpha^2 + \omega_n^2 = 0$$ (7.6)

therefore

$$\alpha = \pm i\omega_n$$ (7.7)

so

$$\phi_n = c_1e^{i\omega_n x} + c_2e^{-i\omega_n x}$$ (7.8)

or

$$\phi_n = A\cos(\omega_n x) + B\sin(\omega_n x)$$ (7.9)

Applying the boundary conditions at $$x = 0$$

$$\phi_n(0) = 0 = A$$ (7.10)

therefore

$$\phi_n = B\sin(\omega_n x)$$ (7.11)

Then, applying the boundary conditions at $$x = l$$

$$\phi_n(l) = 0 = B\sin(\omega_n l)$$ (7.12)

therefore

$$\sin(\omega_n l) = 0 \implies \omega_n = \frac{n\pi}{l}$$ (7.13)

thus

$$\phi_n(x) = \sin\left(\frac{n\pi x}{l}\right)$$ (7.14)

and

$$\lambda_n = k - \frac{n^2\pi^2}{l^2}$$ (7.15)
7.1.3 Inner Product, Orthogonality and Set of Orthogonal Functions

Let us introduce the concept of orthogonal functions. This is based upon the concept of inner product between two real functions, \( f \) and \( g \), of a real variable, \( x \), which is typically defined by\(^2\)

\[
< f, g > = \int_a^b f(x)g(x)dx
\]  
(7.16)

It should be noted that the concept of inner product between functions is closely related to that between vectors. For instance, if the definition of inner product \( < f, g > \) is discretized (using the midpoint formula for the evaluation of an integral) we have

\[
< f, g > \simeq \sum_i f_i g_i \Delta x_i
\]  
(7.17)

whereas the inner product \( < a, b > \) between two vectors \( a \) and \( b \) is defined by

\[
< a, b > = \sum_i a_i b_i
\]  
(7.18)

In analogy to the concept of orthogonality between vectors we define the orthogonality between functions in terms of the inner product: we say that \( f \) and \( g \) are orthogonal in the interval \((a, b)\) if

\[
< f, g > = 0
\]  
(7.19)

The concept of a set of orthonormal functions is closely related to that of a set of orthogonal unit vectors. Let \( \{\phi_j\} \) denote a set of functions \( \phi_j \). Consider a set \( \{\phi_j\} \) of orthogonal functions, that is a set of functions \( \phi_j \) that are mutually orthogonal, \( i.e., \)

\[
< \phi_j, \phi_k > = 0 \quad \text{if} \ j \neq k
\]  
(7.20)

In addition, we assume that the functions have been normalized by imposing the following normalization condition

\[
< \phi_j, \phi_j > = 1
\]  
(7.21)

The last two equations may be combined to yield

\[
< \phi_j, \phi_k > = \delta_{jk}
\]  
(7.22)

\(^2\)This concept can be easily extended for scalar functions of multi-variables. Besides, for two complex functions we have:

\[
< f, g > = \int_a^b f(z)g^*(z)dz
\]
Functions satisfying this equation are called *orthonormal*.

Now, we can generalize the results for a Fourier sine series to any series of orthogonal functions. Consider a set of orthogonal functions \( \{ \phi_j \} \). If a function may be expressed as

\[
f(x) = \sum_j f_j \phi_j(x)
\]

(7.23)

is a *Generalized Fourier Series* where

\[
f_n = \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle}
\]

(7.24)

Observe that the solution is nothing more than a projection of the function \( f \) in the ‘direction’ \( \phi_n \). Here the \( \phi_n \)'s are harmonics each of which is like a vector direction that has been isolated. The \( f_k \) are the magnitudes of the harmonics.

7.1.4 Adjoint and Self-Adjoint Operators – Betti-Castigliano Theorem for Structural Operators

Let us consider an operator \( L \) (including its Boundary Conditions). One can define the *adjoint operator* \( L^* \) (including its Boundary Conditions) in terms of the inner product. By definition, \( L^* \) is the operator such that

\[
\langle Lu, v \rangle = \langle u, L^* v \rangle
\]

(7.25)

An operator is called *self adjoint* if \( L^* = L \), or

\[
\langle Lu, v \rangle = \langle u, Lv \rangle
\]

(7.26)

If one consider a structural operator \( L \), the Betti-Castigliano theorem states that

*the job that the elastic force associated to an arbitrary displacement \( u \) performs on an arbitrary displacement \( v \) is equal to the job that the elastic force associated to an arbitrary displacement \( v \) perform on an arbitrary displacement \( u \)*

Since this statement is formally identical to Eq. 7.26, then the thesis of the Betti-Castigliano theorem is identical to the following statement: *the structural operator is self-adjoint.*

---

3In Section 7.3.1 it will be shown that the Betti-Castigliano theorem is the *integral* form of the property that for an elastic solid there exists locally the functional *elastic potential energy*. 
7.1.4.1 Examples of Self-Adjoint Operators

Suppose we have a linear operator $L$: to see if it is self adjoint we use Eq. 7.26. Next, consider the following 1D and 2D examples

- **Example # 1:** bar operator with clamped boundary conditions
  
  Consider
  \[
  L = \frac{d^2}{dx^2} \quad u(0) = u(l) = 0
  \]  
  (7.27)
  
  Apply the condition $(Lu, v) - (u, Lv) = 0$ then
  \[
  \int_0^l v \left( \frac{d^2 u}{dx^2} \right) dx - \int_0^l u \left( \frac{d^2 v}{dx^2} \right) dx = 0
  \]  
  (7.28)
  
  or
  \[
  \int_0^l \frac{d^2 u}{dx^2} v dx - \frac{d^2 v}{dx^2} u dx = 0
  \]  
  (7.29)
  
  or
  \[
  \int_0^l \frac{d}{dx} \left( v \frac{du}{dx} - u \frac{dv}{dx} \right) dx = 0
  \]  
  (7.30)
  
  thus
  \[
  [vu' - uv']_0^l = 0
  \]  
  (7.31)
  
  \[
  0 = 0
  \]  
  (7.32)

  We conclude that this operator is self-adjoint and then its Eigenfunctions are necessarily orthogonal.

- **Example # 2:** 3-D laplacian with Dirichelet bondary condition
  
  Consider the operator $L = \nabla^2$ with $u = 0$ on edges (Dirichelet’s Boundary Conditions).
  
  Applying the condition given by Eq. 7.26, then
  \[
  \iiint_V (v \nabla^2 u - u \nabla^2 v) \, dxdydz = 0
  \]  
  (7.33)
  
  but
  \[
  \nabla \cdot (f \nabla g) = f \nabla^2 g + \nabla f \cdot \nabla g
  \]  
  (7.34)
  
  Thus,
  \[
  \iiint_V \nabla \cdot (v \nabla u - u \nabla v) \, dxdydz = 0
  \]  
  (7.35)
  
  Applying the Gauss divergence theorem one obtains
  \[
  \iiint_S (v \nabla u - u \nabla v) \cdot nd\sigma = 0
  \]  
  (7.36)
  
  but $u = v = 0$ on $S$ therefore $0 = 0$. Thus the Laplacian is self adjoint.
7.1.5 Theorems on self-adjointness

- If one considers $L\phi_n = \lambda_n \phi_n$ and the corresponding complex conjugate relationship $L\phi^*_n = \lambda^*_n \phi^*_n$ one can obtains:

\[
\langle \phi^*_n, L\phi_n \rangle = \lambda_n \langle \phi^*_n, \phi_n \rangle \tag{7.37}
\]
\[
\langle \phi_n, L\phi^*_n \rangle = \lambda^*_n \langle \phi_n, \phi^*_n \rangle \tag{7.38}
\]

Because of self-adjointness taking the difference one has

\[
0 = (\lambda_n - \lambda^*_n) \langle \phi_n, \phi^*_n \rangle \equiv 2 \text{Im} (\lambda_n) |\phi_n|^2 \tag{7.39}
\]

which is true if and only if $\text{Im} (\lambda_n) = 0$: then, the eigenvalues of a self-adjoint operator are real. As a consequence, the eigenfunctions $\phi_n$ are real too.

- Consider the self-adjoint operator which satisfies the Eigenvalue problem $Lu = \lambda u$. Then since $L^* = L$

\[
\langle L\phi_n, \phi_m \rangle = \lambda_n \langle \phi_n, \phi_m \rangle \tag{7.40}
\]
\[
\langle \phi_n, L\phi_m \rangle = \lambda_m \langle \phi_n, \phi_m \rangle \tag{7.41}
\]

Taking the difference and using Eq. 7.26 we have

\[
\langle \phi_n, \phi_m \rangle (\lambda_n - \lambda_m) = 0 \tag{7.42}
\]

thus

\[
\langle \phi_n, \phi_m \rangle = 0 \quad n \neq m \tag{7.43}
\]

This is the definition of orthogonal functions. Thus, if $L$ is self adjoint the eigenfunctions are orthogonal, i.e., its Eigenfunctions can be used to represent any function in the corresponding domain.

- One defines a self-adjoint operator positive if, for all function $u$, is $\langle u, Lu \rangle > 0$; then, one can obtain (using the orthogonal decomposition for $u$)

\[
0 < \langle Lu, u \rangle = \left( L \sum_n u_n \phi_n, \sum_m u_m \phi_m \right) = \sum_{m,n} u_n u_m \langle L\phi_n, \phi_m \rangle = \sum_{m,n} u_n u_m \lambda_n \langle \phi_n, \phi_m \rangle = \sum_n \lambda_n u_n^2 \tag{7.44}
\]

that is true for an arbitrary function $u$ if and only if $\lambda_n > 0$. 
7.1.6 Eigenvalues and eigenfunction of structural operator v.s. natural frequencies and modes of a structure – Free response

Let us consider in the following the problem of the free response of the structure generally described by the PDE model

\[ Lu + \rho \ddot{u} = 0 \]  \hspace{1cm} (7.45)

homogeneous B.C. \hspace{1cm} (7.46)

\[ u(x, 0) = u_0 \quad \dot{u}(x, 0) = \dot{u}_0 \]  \hspace{1cm} (7.47)

Note that \( L = -EA\partial^2 / \partial x^2 \) for a homogeneous bar, \( L = EI\partial^4 / \partial x^4 \) for a homogeneous beam, \( L = D\nabla^4 \) for a homogeneous plate, whereas \( \rho \) represents a linear (bar and beam) or planar (plate) mass distribution. If \( \phi_n(x) \) and \( \lambda_n \) are the eigenfunctions and the eigenvalues associated to the selfadjoint operator \( L \) and also considering Eq. 7.45, one has:

\[ 0 = Lu + \rho \ddot{u} = L \sum_n \phi_n(x) u_n + \rho \sum_n \ddot{u}_n \phi_n(x) = \sum_n \phi_n(x) (u_n \lambda_n + \rho \ddot{u}_n) = 0 \]  \hspace{1cm} (7.48)

Then, for the independence of the eigenfunctions (indeed, orthogonality) one yields

\[ \rho \ddot{u}_n + \lambda_n u_n = 0 \quad \forall \ n \]  \hspace{1cm} (7.49)

Furthermore, for the initial conditions one can also assume \( u_0(x) = \sum_n u_{0n} \phi_n(x) \) and \( \dot{u}_0(x) = \sum_n \dot{u}_{0n} \phi_n(x) \), and this implies for the orthogonality property and also assuming for simplicity that the eigenfunctions are orthonormalized,

\[ u_{0n}(x) = \langle u_0(x), \phi_n(x) \rangle \quad \dot{u}_{0n}(x) = \langle \dot{u}_0(x), \phi_n(x) \rangle \]  \hspace{1cm} (7.50)

Then, the solution of Eq. 7.49 is given by

\[ u_n(t) = u_{0n} \cos \sqrt{\frac{\lambda_n}{\rho}} t + \frac{\dot{u}_{0n}}{\sqrt{\frac{\lambda_n}{\rho}}} \sin \sqrt{\frac{\lambda_n}{\rho}} t \]  \hspace{1cm} (7.51)

i.e., considering Eq. 7.50,

\[ u(x, t) = \sum_n \phi_n(x) u_n(t) = \sum_n \phi_n(x) \left[ \langle u_0(x), \phi_n(x) \rangle \cos \sqrt{\frac{\lambda_n}{\rho}} t + \langle \dot{u}_0(x), \phi_n(x) \rangle \sqrt{\frac{\lambda_n}{\rho}} \sin \sqrt{\frac{\lambda_n}{\rho}} t \right] \]  \hspace{1cm} (7.52)

Then, if one chooses as initial conditions

\[ u_0(x) = \phi_m(x) \quad \dot{u}_0(x) = 0 \]  \hspace{1cm} (7.53)
one obtain the solution

\[ u(x, t) = \phi_m(x) \cos \sqrt{\frac{\lambda_m}{\varrho}} t \]  \hspace{1cm} (7.54)

which corresponds to the physical definition of natural mode of vibration \( \phi_m(x) \) associated to the natural angular frequency \( \omega_m = \sqrt{\frac{\lambda_m}{\varrho}} \): in fact, if the \( m \)-th mode is imposed as initial condition in a free-response problem (damping free), the solution has a structure given by the same space-function of the mode multiplied by a pure harmonic function of time with (angular) frequency given by the corresponding \( m \)-th natural (angular) frequency.\(^4\) This statement is exactly expressed by Eq. 7.54.

Note also that if one could also defines the generalized eigenfunctions and eigenvalues as given by the solution of the differential problem

\[ L\phi_n(x) = \varrho(x)\lambda_n\phi_n(x) \]  \hspace{1cm} (7.55)

because the positivity of the eigenvalues associated to a positive and selfadjoint operator is not affected by the definition. Nevertheless, the advantage of this definition with respect to the previous one is that the angular natural frequency in this case is directly given by \( \omega_n = \sqrt{\lambda_n} \).

7.1.6.1 Natural modes and frequencies of homogeneous beams

The eigenvalues and eigenfunctions of beams with different boundary conditions are here evaluated. The coordinate along the beam axis is denoted by \( x \), whereas time by \( t \). Space and time derivatives are written as:

\[ (\bullet)' := \frac{d(\bullet)}{dx}, \quad (\bullet) := \frac{d(\bullet)}{dt}. \]

\(^4\)Note that the same procedure can be applied to discrete or discretized systems (see Chapter 6): i.e., also in these cases one can study the eigenproblem associated to the second order linear ordinary differential equation of the structural vibration, and then discover that the eigenvalues and the eigenvector coincide with the physical concepts of natural (angular) frequency and natural mode of vibration of the structure with finite degree of freedoms.
The following quantities are also introduced:

\[ l = \text{beam length} \]
\[ E(x) = \text{Young’s modulus of the material} \]
\[ \rho(x) = \text{density of the material} \]
\[ A(x) = \text{area of the beam cross-section} \]
\[ I(x) = \text{moment of inertia of the beam cross-section} \]
\[ w(x; t) = \text{vertical displacement of the elastic line} \]
\[ \lambda_n = n\text{-th beam eigenvalue} \]
\[ \phi_n(x) = n\text{-th beam eigenfunction} \].

Once the assumptions \( \rho A(x) = \text{const} \), \( EI(x) = \text{const} \) are made, the governing equation for the beam free response is written as:

\[ EIw'''' + \rho A\ddot{w} = 0 \tag{7.56} \]

with the initial conditions:

\[ w(x; 0) = w_0(x), \quad \dot{w}(x; 0) = \dot{w}_0(x). \]

The problem is completed with the boundary conditions at the beam ends (\( x = 0, l \)).

The beam eigenvalues \( \lambda_n > 0 \) and eigenfunctions \( \phi_n \) satisfy the differential problem \( L\phi_n = \lambda_n\phi_n \) with the boundary conditions, where \( L(\bullet) = EI(\bullet)'''' \) is the linear structural differential beam operator. In the present case, one obtains:

\[ \phi_n'''' - \frac{\lambda_n}{EI}\phi_n = 0. \tag{7.57} \]

Once the positive quantity \( m_n^4 := \lambda_n/(EI) \) is introduced, the characteristic equation associated to Eq. (7.57) is written as \( \alpha^4 - m_n^4 = 0 \), which has the roots \( \alpha_{1,2} = \pm m_n \) and \( \alpha_{3,4} = \pm jm_n \). The eigenfunctions are then searched in the general form:

\[ \phi_n(x) = c_1e^{\alpha_1x} + c_2e^{\alpha_2x} + c_3e^{\alpha_3x} + c_4e^{\alpha_4x} = A_n \cos(m_nx) + B_n \sin(m_nx) + C_n \cosh(m_nx) + D_n \sinh(m_nx), \tag{7.58} \]

which yields:

\[ \phi_n'(x) = m_n [-A_n \sin(m_nx) + B_n \cos(m_nx) + C_n \sinh(m_nx) + D_n \cosh(m_nx)], \]
\[ \phi_n''(x) = m_n^2 [-A_n \cos(m_nx) - B_n \sin(m_nx) + C_n \cosh(m_nx) + D_n \sinh(m_nx)], \tag{7.59} \]
\[ \phi_n'''(x) = m_n^3 [A_n \sin(m_nx) - B_n \cos(m_nx) + C_n \sinh(m_nx) + D_n \cosh(m_nx)]. \]
In the following subsections, the computation of the beam eigenvalues and eigenfunctions is specialized to the cases of a simply-supported and of a cantilevered beam.

Once the eigenvalues and eigenfunctions of the beam with given boundary conditions are evaluated, the vertical displacement of the elastic line can be written as:

\[ w(x; t) = \sum_{n=1}^{\infty} \phi_n(x)w_n(t), \]  

(7.60)

which substituted into Eq. (7.56) gives:

\[ L \sum_{n=1}^{\infty} \phi_n w_n + \rho A \sum_{n=1}^{\infty} \phi_n \ddot{w}_n = 0. \]

Using \( L\phi_n = \lambda_n\phi_n \) one has:

\[ \sum_{n=1}^{\infty} \phi_n (\lambda_n w_n + \rho A \ddot{w}_n) = 0, \]  

(7.61)

which yields the uncoupled ordinary differential equations:

\[ \ddot{w}_n + \frac{\lambda_n}{\rho A} w_n = 0 \quad \text{with} \quad n = 1, 2, \ldots, \infty. \]  

(7.62)

These are the governing equations of a set of uncoupled harmonic oscillators of natural angular frequencies given by \( \omega_n = \sqrt{\lambda_n/(\rho A)}. \)

**Simply-supported beam**

The eigenvalues and eigenfunctions of a simply-supported beam are here evaluated. In the present case, the boundary conditions are written as:

\[ w(0; t) = 0, \quad w(l; t) = 0, \quad w''(0; t) = 0, \quad w''(l; t) = 0, \]

which implies:

\[ \phi_n(0, l) = 0, \quad \phi''_n(0, l) = 0. \]  

(7.63)

Using Eq. (7.58) and (7.59), Eq. (7.63) yields:

\[ +A_n + C_n = 0, \]

\[ -A_n + C_n = 0, \]

\[ +A_n \cos(m_n l) + B_n \sin(m_n l) + C_n \cosh(m_n l) + D_n \sinh(m_n l) = 0, \]

\[ -A_n \cos(m_n l) - B_n \sin(m_n l) + C_n \cosh(m_n l) + D_n \sinh(m_n l) = 0. \]

The first two relations imply \( A_n = 0, C_n = 0, \) so that the last two conditions are recast as:

\[ \begin{bmatrix} +\sin(m_n l) & +\sinh(m_n l) \\ -\sin(m_n l) & +\sinh(m_n l) \end{bmatrix} \begin{bmatrix} B_n \\ D_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \]  

(7.64)
In order to have a non-trivial solution, the determinant of the coefficient matrix has to vanish:

\[ 2 \sin(mn) \sinh(mn) = 0 , \]

which gives \( mn = n\pi/l \) with \( n = 1, 2, \ldots, \infty \). Thus, the eigenvalues are evaluated as:

\[ \lambda_n = EI \left( \frac{n\pi}{l} \right)^4 \quad \text{with} \quad n = 1, 2, \ldots, \infty . \tag{7.65} \]

and therefore

\[ \omega_n = \sqrt{\frac{EI}{\rho A} \left( \frac{n\pi}{l} \right)^2} . \tag{7.66} \]

Finally, the first equation in the system (7.64) gives:

\[ D_n = -B_n \frac{\sin(mn)}{\sinh(mn)} , \]

which using the condition of vanishing determinant yields \( D_n = 0 \). Thus, the eigenfunctions of the simply-supported beam are written as:

\[ \phi_n(x) = B_n \sin \left( \frac{n\pi x}{l} \right) \quad \text{with} \quad n = 1, 2, \ldots, \infty . \tag{7.67} \]

Cantilevered beam

The eigenvalues and eigenfunctions of a cantilevered beam are here evaluated. In the present case, the boundary conditions are written as:

\[ w(0; t) = 0 , \quad w'(0; t) = 0 , \quad w''(l; t) = 0 , \quad w'''(l; t) = 0 , \]

which implies:

\[ \phi_n(0) = \phi'_n(0) = 0 , \quad \phi''_n(l) = \phi'''_n(l) = 0 . \tag{7.68} \]

Using Eq. (7.58) and (7.59), Eq. (7.68) yields:

\[ +A_n + C_n = 0 , \]
\[ +B_n + D_n = 0 , \]
\[ -A_n \cos(mn) + B_n \sin(mn) + C_n \cosh(mn) + D_n \sinh(mn) = 0 , \]
\[ +A_n \sin(mn) + B_n \cos(mn) + C_n \sinh(mn) + D_n \cosh(mn) = 0 . \]

The first two relations gives \( C_n = -A_n \), \( D_n = -B_n \), so that the last two conditions are recast as:

\[ \begin{bmatrix} + \cos(mn) & \cosh(mn) \\ + \sin(mn) & \sinh(mn) \end{bmatrix} \begin{bmatrix} A_n \\ B_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} . \tag{7.69} \]
In order to have a non-trivial solution, the determinant of the coefficient matrix has to vanish:

\[- [\cos(mnl) + \cosh(mnl)]^2 - [\sin^2(mnl) - \sinh^2(mnl)] = 0,
\]

which can be recast as:

\[\cos(mnl) \cosh(mnl) = -1.\]

A graphical representation of the above condition is illustrated in Fig. 7.1. The roots are identified as the intersections between the two curves. The first two roots are evaluated as \(m_1 = 1.875/l\) and \(m_2 = 4.694/l\), whereas the following ones can be approximated as:

\[m_n = (2n - 1) \frac{\pi}{2l} \quad \text{with} \quad n = 3, 4, \ldots, \infty.\]

Thus, the eigenvalues are evaluated as:

\[
\begin{align*}
\lambda_1 &= EI \left( \frac{1.875}{l} \right)^4, \\
\lambda_2 &= EI \left( \frac{4.694}{l} \right)^4, \\
\lambda_n &= EI \left[ (2n - 1) \frac{\pi}{2l} \right]^4
\end{align*}
\]

(7.70)

or

\[
\begin{align*}
\omega_1 &= \sqrt{\frac{EI}{\rho A}} \left( \frac{1.875}{l} \right)^2, \\
\omega_2 &= \sqrt{\frac{EI}{\rho A}} \left( \frac{4.694}{l} \right)^2, \\
\omega_n &= \sqrt{\frac{EI}{\rho A}} \left[ (2n - 1) \frac{\pi}{2l} \right]^2.
\end{align*}
\]

(7.71)

Finally, the second equation in the system (7.69) gives:

\[B_n = A_n \sin(mnl) - \sinh(mnl) \cos(mnl) + \cosh(mnl) = A_n \overline{A}_n,
\]

so that \(D_n = -A_n \overline{A}_n\). Thus, the eigenfunctions of the cantilevered beam are written as:

\[
\phi_n(x) = A_n \left[ \cos(mnl) + \overline{A}_n \sin(mnl) - \cosh(mnl) - \overline{A}_n \sinh(mnl) \right],
\]

(7.72)

where \(A_n\) is an arbitrary constant.

### 7.1.7 Static and Dynamic structural problems using the eigenfunction method and influence function method – Static and Dynamic responses

A general procedure for solving PDE problem using the eigenfunction method end the influence function or Green function method will be presented.

Only for the sake of clarity, the eigenfunction will be assumed to be ortho-normalized, \(i.e.,\) such as

\[
\langle \phi_n(x'), \phi_n(x') \rangle := \iiint_{\cal V} \phi_n(x') \phi_n(x') dx' = 1
\]

(7.73)

where \(\cal V\) is the generic space domain where the problem is defined with a boundary equal to \(S := \partial \cal V\).
7.1.7.1 Static problems using the eigenfunction method (Navier’s approach)

Consider the space-differential problem

\[ Lu = f \]  

(7.74)

together with homogeneous boundary conditions. If there exists a set of eigenfuncions \( \phi_n(x) \) and eigenvectors \( \lambda_n \) for the operator \( L \) one could assume \( u = \sum_n u_n \phi_n(x) \) and then

\[ Lu = L \sum_n u_n \phi_n = \sum_n u_n L \phi_n = \sum_n u_n \lambda_n \phi_n = f = \sum_n f_n \phi_n \]  

(7.75)

If the eigenfunctions \( \phi_n \) are independent,\(^6\) one has \( u_n \lambda_n = f_n \) and then

\[ u = \sum_n \frac{f_n}{\lambda_n} \phi_n(x) \]  

(7.76)

which is the static solution with the eigenfunction method.\(^7\)

---

\(^5\)The unicity of such decomposition has been proved if \( L \) is a self-adjoint operator.

\(^6\)Note that if the operator is self-adjoint, the eigenfunctions are orthogonal and then \( f_n = \langle f, \phi_n \rangle \).

\(^7\)The above procedure is identical to that used in the linear algebra for solving the problem \( Ax = b \). Indeed, if one considers the corresponding eigenproblem \( A z^{(n)} = \lambda_n z^{(n)} \) (for the sake of simplicity the hypothesis of distinct eigenvalues has been assumed), one can represent vector \( x \) as \( x = \sum_1^N \alpha_n z^{(n)} \) and vector \( b \) as \( b = \sum_1^N \beta_n z^{(n)} \); then

\[ A x = A \sum_1^N \alpha_n z^{(n)} = \sum_1^N \alpha_n A z^{(n)} = \sum_1^N \alpha_n \lambda_n z^{(n)} = b = \sum_1^N \beta_n z^{(n)} \]
7.1.7.2 Static problems using the Green function method

The final formula of the Green function method can be directly obtained from Eq. 7.76 of the eigenfunction method: indeed, since in the case of orthonormalized eigenfunction (i.e., \( \langle \phi_n(x), \phi_n(x) \rangle = 1 \)), \( f_n := \langle f(x), \phi_n(x) \rangle \), Eq. 7.76 yields

\[
    u(x) = \sum_{n} \frac{\langle f(x'), \phi_n(x') \rangle}{\lambda_n} \phi_n(x) = \left\langle \sum_{n} \frac{\phi_n(x)\phi_n(x')}{\lambda_n}, f(x') \right\rangle = \iiint_{\mathcal{V}} G(x, x') f(x') \, dx' \quad (7.77)
\]

where

\[
    G(x, x') := \sum_{n} \frac{\phi_n(x)\phi_n(x')}{\lambda_n} \quad (7.78)
\]

is the influence or Green function of the problem given by Eq. 7.74: the interpretation of \( G(x, x') \) as influence function is immediate if one assume \( f(x') = \delta(x' - x^*) \) in Eq. 7.77. Furthermore, if one symbolically inverted Eq. 7.74 as \( u = L^{-1} f \), then the inverse operator would be given by

\[
    L^{-1} \bullet := \iiint_{\mathcal{V}} G(x, x') \bullet \, dx' \quad (7.79)
\]

However, the same result could be obtained introducing formally the space Green function \( G \) as solution of the adjoint problem

\[
    L^* G = \delta(x - x') + \text{homogeneous B.C.} \quad (7.80)
\]

where \( L^* \) is the adjoint operator of \( L \). Now, let us assume for a while that the solution of Eq. 7.80 is given: then taking the internal product of Eq. 7.74 multiplied by \( G \) and of Eq. 7.80 multiplied by \( u \) one obtains

\[
    \langle G, Lu \rangle = \langle G, f \rangle \quad (7.81)
\]
\[
    \langle u, L^* G \rangle = \langle u, \delta(x - x') \rangle \quad (7.82)
\]

Then, taking the difference and using the definition of adjoint operator one has\(^8\)

\[
    u(x') = \langle G(x, x'), f(x) \rangle \quad (7.83)
\]

\*Note that if the Boundary Conditions for the original problem and the adjoint problem were not homogeneous, this difference may yield boundary condition terms in the right-hand-side of Eq. 7.83 strictly dependent by the definition of the adjoint problem.

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The Green function $G$ will be determined in the following applying the eigenfunction method to the problem given by Eq. 7.80 and assuming that the operator $L^*$ is selfadjoint. Then, the solution for $G$ can be assumed to be of the form

$$G(x, x') = \sum_{n} g_n(x') \phi_n(x) \quad (7.84)$$

and so

$$L^* G = \sum_{n} g_n(x') L \phi_n(x) = \sum_{n} g_n(x') \lambda_n \phi_n(x) = \delta(x - x') = \sum_{n} \delta \phi_n(x) \quad (7.85)$$

where $\delta_n = \langle \delta(x - x'), \phi_n(x) \rangle \equiv \phi_n(x')$. From the independence of the eigenfunctions one has $g_n = \delta_n / \lambda_n = \phi_n(x') / \lambda_n$, i.e., from Eq. 7.84,

$$G(x, x') := \sum_{n} \frac{\phi_n(x) \phi_n(x')}{\lambda_n} \quad (7.86)$$

$G$ represents the response in $x$ to a load concentrated at point $x'$; if one should consider a generic load condition given by the function $f(x)$ the response would be given by Eq. 7.83 as combination of impulsive response function.

### 7.1.7.3 Dynamic problems using the eigenfunction method

Let us consider again the general problem given by Eqs. 7.100, 7.101, and 7.101: if there exists a set of eigenfunctions $\phi_n(x)$ and eigenvector $\lambda_n$ for the operator $L$ one could assume $u(x, t) = \sum_{n} u_n(t) \phi_n(x)$ and then

$$L_d u(x, t) = \sum_{n} (u_n(t) L \phi_n(x)) + \sum_{m} (\rho \ddot{u}_m(t) \phi_m(x)) = \sum_{n} \phi_n(x) (\lambda_n u_n(t) + \rho \ddot{u}_n(t))$$

$$= f(x, t) = \sum_{n} f_n(t) \phi_n(x) \quad (7.87)$$

As the eigenfunctions are independent, one has

$$\rho \ddot{u}_n + \lambda_n u_n = f_n \quad (7.88)$$

$$u_n(0) = u_{0_n}, \quad \dot{u}_n(0) = \dot{u}_{0_n} \quad (7.89)$$

with, assuming the orthonormality of the eigenfunctions, i.e., $\langle \phi_n(x), \phi_n(x) \rangle = 1$,

$$u_{0_n} = \langle u(x, 0), \phi_n(x) \rangle \quad (7.90)$$

$$\dot{u}_{0_n} = \langle \dot{u}(x, 0), \phi_n(x) \rangle \quad (7.91)$$

$$f_n(t) = \langle f(x, t), \phi_n(x) \rangle \quad (7.92)$$
The solution of the problem given by Eqs. 7.88 and 7.89 is

\[ u_n(t) = \int_0^t \frac{\sin \omega_n (t - \tau)}{\rho \omega_n} f_n(\tau) d\tau + u_0 n \cos \omega_n t + \frac{\dot{u}_0 n}{\omega_n} \sin \omega_n t \]  

(7.92)

Then, the final solution, considering Eqs. 7.90, 7.91, and 7.92, is

\[ u(x, t) = \sum_n \phi_n(x) \left\{ \int_0^t \langle f(x, \tau), \phi_n(x) \rangle h_n(t - \tau) d\tau + \langle \dot{u}(x, 0), \phi_n(x) \rangle h_n(t) + \langle u(x, 0), \phi_n(x) \rangle \dot{h}_n(t) \right\} \]  

(7.93)

where \( h_n(t) := \sin \omega_n t / \rho \omega_n \) is again the impulsive response of an elementary undamped mechanical system.\(^9\)

The above equation allows, in the hypothesis of initial condition equal to zero, to find the relationship between the Fourier transform \( \tilde{f}(\bar{x}, \omega) \) of the input and the Fourier transform of the output equal to \( \tilde{u}(x, \omega) \); i.e.,

\[ \tilde{u}(x, \omega) = \sum_n \phi_n(x_*) \frac{\langle \tilde{f}(\bar{x}, \omega), \phi_n(x) \rangle}{\rho (\lambda_n - \omega^2)} \]  

(7.94)

Note that if a finite number of structure points are fixed for the input e.g., the case of so-called "pin force" located at the point \( x_* \) with a constant spectrum equal to 1

\[ \tilde{f}(\bar{x}, \omega) = \delta(\bar{x} - x_*) \]  

(7.95)

the Eq. 7.96 can define analytically Frequency Responce Functions (FRF's) relative to the input point \( x_* \) and the output point \( x \), or

\[ \tilde{u}(x_*, \omega) = \sum_n \phi_n(x_*) \phi_n(x) \frac{\delta(\bar{x} - x_*)}{\rho (\lambda_n - \omega^2)} \]  

(7.96)

Furthermore, note that if one consider a simple harmonic load with angular frequency close to a generic natural angular frequency \( \sqrt{\lambda_n} \), then a resonance condition for the \( n \)-th mode is induced.

In the following a simple application of the eigenfunctions methos is presented trying to show a parallelism between this approach (for space in infinitive dimension) and the eigenvector approach (for space in finite dimension).

\( ^9 \)This result is identical to that obtained with the Green function method by Eq. 7.132 in the same hypothesis of homogeneous boundary condition.
Infinitive dimension

\[ \ddot{u} + Lu = f \] with \( L = L^* \)
c.c. = 0
\[ u(x, 0) = h_0(x) \]
\[ \dot{u}(x, 0) = h_1(x) \]

\[ \ddot{u} + \omega_n^2 u_n = f_n \]
setting:
\[ u = \sum_{n=1}^{\infty} u_n \phi_n \] with \( L\phi_n = \omega_n^2 \phi_n \) and B.C. = 0
\[ f_n = \frac{<\phi_n, f>}{<\phi_n, \phi_n>} \] or \( f = \sum_{n=1}^{\infty} f_n \phi_n \)
\[ u_n(0) = \frac{<\phi_n, h_0>}{<\phi_n, \phi_n>} \] or \( h_0 = \sum_{n=1}^{\infty} u_n(0) \phi_n \)
\[ \dot{u}_n(0) = \frac{<\phi_n, h_1>}{<\phi_n, \phi_n>} \] or \( h_1 = \sum_{n=1}^{\infty} \dot{u}_n(0) \phi_n \)

Finite dimension

\[ \ddot{x} + Ax = f \] con \( A = A^T \)
\[ x(0) = h_0 \]
\[ x(0) = h_1 \]
\[ \ddot{y}_n + \omega_n^2 y_n = g_n \]
setting:
\[ x = \sum_{n=1}^{N} y_n z_n \] with \( Az_n = \omega_n^2 z_n \)
\[ g_n = \frac{<z_n, f>}{<z_n, z_n>} \] or \( f = \sum_{n=1}^{N} g_n z_n \)
\[ y_n(0) = \frac{<z_n, h_0>}{<z_n, z_n>} \] or \( h_0 = \sum_{n=1}^{N} y_n(0) z_n \)
\[ \dot{y}_n(0) = \frac{<z_n, h_1>}{<z_n, z_n>} \] or \( h_1 = \sum_{n=1}^{N} \dot{y}_n(0) z_n \)

Then, for the following problem

\[ \ddot{u} - u_{xx} = \sin \frac{3\pi x}{\ell} \cos \Omega t \]
\[ u(0, t) = u(\ell, t) = 0 \]
\[ u(x, 0) = 5 \sin \frac{5\pi x}{\ell} \]
\[ \dot{u}(x, 0) = 7 \sin \frac{7\pi x}{\ell} \]

one has as eigenfunctions \( \phi_n = \sqrt{\frac{2}{\ell}} \sin \frac{n\pi x}{\ell} \) and eigenvalues \( \omega_n = \frac{n\pi}{\ell} \) and

\[ f_n := \begin{cases} 
\cos \Omega t & \text{for } n = 3 \\
0 & \text{in other cases}
\end{cases} \quad (7.97) \]

\[ u_n(x, 0) := \begin{cases} 
5 & \text{for } n = 5 \\
0 & \text{in other cases}
\end{cases} \quad (7.98) \]

\[ \dot{u}_n(x, 0) := \begin{cases} 
7 & \text{for } n = 7 \\
0 & \text{in other cases}
\end{cases} \quad (7.99) \]

Then, one has three uncoupled problems:

1. \( \ddot{u}_3 + \omega_3^2 u_3 = \cos \Omega t \) with \( u_3 = \dot{u}_3 = 0 \)
2. \( \ddot{u}_5 + \omega_5^2 u_5 = 0 \) with \( u_5 = 5; \dot{u}_5 = 0 \)
3. \( \ddot{u}_7 + \omega_7^2 u_7 = 0 \) with \( u_7 = 0; \dot{u}_7 = 7 \)
whereas all the other \( u_n \) are identically equal to zero. Thus, the final solution is:

\[
u(x, t) = \sqrt{\frac{2}{\ell}} \left[ \frac{1}{\omega_3^2 - \Omega^2} (\cos \Omega t - \cos \omega_3 t) \sin \frac{3\pi x}{\ell} + 5 \cos \omega_5 t \sin \frac{5\pi x}{\ell} + \frac{7}{\omega_7^2} \sin \omega_7 t \sin \frac{7\pi x}{\ell} \right]
\]

### 7.1.7.4 Dynamic problems using the Green function method

Let us consider the linear PDE problem\(^{10}\)

\[
Lu + \ddot{u} = f \quad \text{(7.100)}
\]

homogeneous B.C. \( \quad \text{(7.101)} \)

arbitrary I.C. \( t \in (0, T) \) \( \quad \text{(7.102)} \)

where \( L \) is the structural positive self-adjoint operator introduced before.

It should be worth to note that a global dynamic operator could be introduced

\[
L_d \bullet := (L + \partial^2 / \partial t^2) \bullet.
\]

Then, defining an internal product in space and in time, \( \langle \cdot, \cdot \rangle_{xt} \), one could introduce the following concept of self-adjoint operator \( L_d^* \)

\[
\langle v, L_d u \rangle_{xt} - \langle u, L_d v \rangle_{xt} = \text{space and time boundary terms} \quad \text{(7.103)}
\]

where \( u \) and \( v \) are the standard trial functions, space boundary terms are present if the problem boundary conditions are not homogeneous, whereas the time boundary terms are always present if one has a typical initial value problem. This issues can be simply clarified with the following

**Example**

Consider the linear PDE problem two-dimensional in space

\[
L_d \bullet := \nabla^2 \bullet + \frac{\partial^2 \bullet}{\partial t^2} \quad \text{(7.104)}
\]

Then one can obtain

\[
vL_d u - uL_d v = v(\nabla^2 u + \ddot{u}) - u(\nabla^2 v + \ddot{v}) = \nabla \cdot (v \nabla u - u \nabla v) + (v\ddot{u} - u\ddot{v}) \quad \text{(7.105)}
\]

i.e., developing the left-hand side of Eq. 7.103 and considering the Gauss theorem applied on the space integral on the 2-D domain \( A \), one has

\[
\langle v, L_d u \rangle_{xt} - \langle u, L_d v \rangle_{xt} = \int_0^T \int_A \left( v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) ds dt + \int_0^T \int_A (v\ddot{u} - u\ddot{v}) \, dA \quad \text{(7.106)}
\]

which shows that a **space** boundary terms may exist in the definition if one has non-homogeneous Boundary Conditions \( u \neq 0, \ v \neq 0, \ \partial u / \partial n \neq 0, \) or \( \partial v / \partial n \neq 0 \), whereas there is always a **time** boundary term if one has prescribed initial condition only.

The general procedure of the Green function method for dynamic problem can be organized in the following steps.

---

\(^{10}\) As pointed out in Section 7.1.6, the presence of the density function \( \varrho \) is not essential.
1. A \textit{formally} adjoint operator $L_d^*$ can be defined in the sense clarified above and this implies that

$$\langle v, L_d u \rangle_{xt} - \langle u, L_d v \rangle_{xt} = \text{space and time boundary terms} \quad (7.107)$$

2. Let us define a Green space-time function $G$ as solution of the problem

$$L_d^*G(x, x^*; t, t^*) = \delta(x - x^*)\delta(t - t^*) \quad (7.108)$$

The function $G$ will have the role of $v$ in the Eq. 7.107 and it will satisfy boundary, initial and final conditions to be defined.

3. Then, taking the generalized inner product (in space and time) between Eq. 7.100 and $G$, taking the same product between Eq. 7.108 and $u$, and then performing the difference, one can obtain the explicit form of Eq. 7.107

$$\int_0^T \iiint_V G(x, x^*; t, t^*) f(x, t) dV dt - u(x^*, t^*) =$$

$$\int_0^T \iiint_S \text{flux } dS \ dt + \iiint_V \left( \dot{u}G(x, x^*; t, t^*) - u\dot{G}(x, x^*; t, t^*) \right) \Bigg|_0^T dV$$

where $V$ represents the generic space domain and $S$ its boundary. Note that with the term “flux” we have generically indicated non-homogeneous boundary condition terms for the solution $u$ and the Green function $G$ that, after a standard application of the Gauss theorem, can be typically expressed in a flux form (see the example at the beginning of this Section). However, if one assume to have \textit{homogeneous} boundary condition either for the original problem or for the adjoint or Green function problem (Eq. 7.108), this boundary terms are identically equal to zero: this assumption - which is in general not necessary in the Green function method - will be done in the following. The Equation 7.109 becomes

$$u(x^*, t^*) = \int_0^T \iiint_V G(x, x^*; t, t^*) f(x, t) dV dt$$

$$- \iiint_V \left( G(x, x^*; t, t^*) \dot{u} - u\dot{G}(x, x^*; t, t^*) \right) \Bigg|_0^T dV + \iiint_V \left( G(x, x^*; t, t^*) \ddot{u} - u\ddot{G}(x, x^*; t, t^*) \right) \Bigg|_0^T dV$$

4. It is interesting to observe that the initial or final condition on $G$ are arbitrary but they are not yet established. However, only the \textit{initial} condition on $u$ and $\dot{u}$ are known in the standard dynamic problem. Then, observing the structure of Eq. 7.110, one can easily argue that the more convenient choice for the final condition of $G$ is

$$G(x, x^*; T, t^*) \equiv \dot{G}(x, x^*; T, t^*) \equiv 0$$

$$\quad (7.111)$$
in order to rewrite Eq. 7.110 as
\[
u(x, t) = \int_0^T \int_{\Omega} G(x, x^*; t, t_*) f(x, t) d V dt + \int_0^T \int_{\Omega} (G(x, x^*; t, t_*) \dot{u} - u \dot{G}(x, x^*; t, t_*)) d V dt + \int_{\Omega} G(x, x^*; t, t_*) d V (7.112)
\]

Note that Eq. 7.112 is the final equation of the Green function method: however, the solution should be complete after determining the Green function \( G \) which is the final point in the next item.

5. As we discussed before, the problem for determining the function \( G \) is given by Eq. 7.108 with the following conditions

\begin{align*}
\text{homogeneous space Boundary Conditions} & \quad (7.113) \\
G(x, x^*; T, t_*) & \equiv \dot{G}(x, x^*; T, t_*) \equiv 0 (7.114)
\end{align*}

The problem of the identification of the Green function will be solved in the following with the eigenfunction method (see next Section), i.e., it can be expressed in term of the eigenfunctions associated to the structural operator \( L \):

\[
G(x, x^*; t, t_*) = \sum_{n} G_n(x^*; t, t_*) \phi_n(x) (7.115)
\]

Then, using Eqs. 7.108 and 7.115 and considering that the operator \( L_d \) is formally selfadjoint, one has

\[
L_d^* G(x, x^*; t, t_*) = L_d \sum_{n} \int_{\Omega} G_n(x^*; t, t_*) \phi_n(x) = \sum_{n} \left[ L + \frac{\partial^2}{\partial t^2} \right] (G_n(x^*; t, t_*) \phi_n(x)) (7.116)
\]

\[
+ \sum_{n} \left[ \lambda_n G_n(x^*; t, t_*) + \dot{G}_n(x^*; t, t_*) \right] \phi_n(x) = \delta(x - x^*) \delta(t - t_*) = \sum_{n} \delta(t - t_*) \phi_n(x) \phi_n(x^*)
\]

or, for the independence of the eigenfunctions \( \phi_n(x^*) = \int_{\Omega} \delta(x - x^*) \phi_n(x) d V \),

\[
\ddot{G}_n + \lambda_n G_n = \delta(t - t_*) \phi_n(x) (7.117) \quad G_n(T) = 0 \quad \dot{G}_n(T) = 0 (7.118)
\]

where a more syntetic notation has been introduced and where the final conditions have been obtained by Eq. 7.114 by the spectral decomposition

\[
G(T) = \sum_{n} G_n(T) \phi_n(x) = 0 \quad \dot{G}(T) = \sum_{n} \dot{G}_n(T) \phi_n(x) = 0 (7.119)
\]

Now, the final value problem given by Eqs. 7.117 and 7.118 is not so easily integrable, whereas the initial value problem:

\[
\ddot{g}_n^* + \lambda_n g_n^* = \delta(t - t_*) (7.120) \quad g_n^*(0) = 0 \quad \dot{g}_n^*(0) = 0 \quad (t_* > 0) (7.121)
\]
has the elementary solution
\[ g_n^*(t) = H(t-t_*) \frac{1}{\sqrt{\lambda_n}} \sin \sqrt{\lambda_n}(t-t_*) \] (7.122)

The next step is to acknowledge that the function \( g_n^* \) is reciprocal function in the sense of the reciprocity theorem (Lanczos) of \( g_n \) defined by the problem
\[ \ddot{g}_n + \lambda_n g_n = \delta(t-t_*) \] (7.123)
\[ g_n(T) = 0 \quad \dot{g}_n(T) = 1 \] (7.124)

Indeed, the reciprocity theorem states that

If there exist two function \( M \) and \( N \) such that
\[ L_d M = \delta(x-x_1)\delta(t-t_1) \quad \text{and} \quad L_d^* N = \delta(x-x_2)\delta(t-t_2) \] (7.125)

with boundary terms all equal to zero, then
\[ M(x_1,x_2;t_1,t_2) = N(x_2,x_1;t_2,t_1) \] (7.126)

The proof can be easily demonstrated by the definition of self-adjoint that implies
\[ \langle M, L_d^* N \rangle_{x_t} - \langle N, L_d^* M \rangle_{x_t} = 0 \] (7.127)

or
\[ \int_0^T \int_V M(x,x_1;t_1)\delta(x-x_2)\delta(t-t_2)dVdt = \int_0^T \int_V N(x,x_2;t_2)\delta(x-x_1)\delta(t-t_1)dVdt \]
which demonstrates the thesis.

Now, we can verify that the function \( g_n^* \) and \( g_n \) satisfy the hypothesis of the reciprocity theorem only for the time part, i.e., considering on the time operator \( L_t \bullet := d^2 \bullet / dt^2 + \lambda_n \bullet \) and the correspondig inner product \( \langle \bullet, \nabla \rangle_t \). One has
\[ \langle g_n^*, L_t^* g_n \rangle_t - \langle g_n, L_t g_n^* \rangle_t = \int_0^T g_n^* (\ddot{g}_n + \lambda_n g_n) dt - \int_0^T g_n (\ddot{g}_n^* + \lambda_n g_n^*) dt = \int_0^T (g_n^* \ddot{g}_n - g_n \ddot{g}_n^*) dt = (g_n^* \ddot{g}_n - g_n \ddot{g}_n^*)|_T - (g_n^* \ddot{g}_n - g_n \ddot{g}_n^*)|_0 = 0 \]

\[ \text{Note also that the Eqs. 7.120 and 7.121 can be rewritten as} \]
\[ \ddot{g}_n + \lambda_n g_n = 0 \]
\[ g_n(t_*) = 0 \quad \dot{g}_n(t_*) = 1 \]
which is identically equal to zero for homogeneous initial conditions chosen for \( g^*_n \) (Eq. 7.121) and for the homogeneous final conditions chosen for \( g_n \) (Eq. 7.124). As the hypothesis of the reciprocity theorem is satisfied, the thesis of the theorem then implies (see Eq. 7.122)

\[
g_n(t) = H(t_\ast - t) \frac{1}{\sqrt{\lambda_n}} \sin \sqrt{\lambda_n}(t_\ast - t) \tag{7.128}
\]

Furthermore, as the linear problem given by Eqs. 7.117 and 7.118 is identical to the linear problem given by Eqs. 7.123 and 7.124 but a constant \( \phi_n(x_\ast) \), then one obtains

\[
G_n(t) = H(t_\ast - t) \frac{1}{\sqrt{\lambda_n}} \sin \sqrt{\lambda_n}(t_\ast - t) \phi_n(x_\ast) \tag{7.129}
\]

and finally, see Eq. 7.115,

\[
G(x, x_\ast; t, t_\ast) = \sum_n H(t_\ast - t) \frac{1}{\sqrt{\lambda_n}} \sin \sqrt{\lambda_n}(t_\ast - t) \phi_n(x_\ast) \phi_n(x) \tag{7.130}
\]

Note the obtained time-space Green function is not causal in time; nevertheless, it is not the solution of the original problem (which is an initial value problem) but it is solution of the adjoint one which is a final value problem.

Now, the final formula for the solution can be obtained using Eq. 7.112 together with the expression of the Green function given by Eq. 7.130:

\[
u(x_\ast, t_\ast) = \int_0^{t_\ast} \int \int_V \left[ \sum_n \frac{1}{\sqrt{\lambda_n}} \sin \sqrt{\lambda_n}(t_\ast - t) \phi_n(x_\ast) \phi_n(x) \right] f(x, t) dV dt \tag{7.131}
\]

or,

\[
u(x_\ast, t_\ast) = \sum_n \phi_n(x_\ast) \left\{ \int_0^{t_\ast} \langle f(x, t), \phi_n(x) \rangle h_n(t_\ast - t) dt + \langle \dot{u}(x, 0), \phi_n(x) \rangle h_n(t_\ast) + \langle u(x, 0), \phi_n(x) \rangle \dot{h}_n(t_\ast) \right\}
\]

where \( h_n(t) := \sin \sqrt{\lambda_n} t / \sqrt{\lambda_n} \) is the impulsive response of an elementary undamped mechanical system. It has to be pointed out that this solution is valid only for homogeneous boundary conditions: a more general formula is given by Eq. 7.109. Indeed, the assumptions of homogeneous boundary conditions on for \( u \), and then for \( G \), have allowed to use the eigenfunctions method to determine the time-space Green function.\(^\text{12}\)

\(^\text{12}\)For generic boundary conditions the Green function could not be easily found analytically; a further possibility (see Section 7.3) can be to find analytically a Green function solution that does not satisfy any boundary condition, i.e., a “free-space” Green function. This approach, which is typical in fluidynamics, implies that boundary terms involving \( G \) and the unknown \( u \) are present in the integral representation of the solution.
7.2 Discretized Eigenfunction methods: comparisons with other standard discretization methods

In the following subsections the eigenfunction methods will be considered with a finite (or truncated) number of eigenfunctions as typically done in the space-discretization approach in structures. Then, some comparisons with standard discretization method as Galerkin method, Rayleigh-Ritz method, and Finite-Element method together with some general theoretical comments will be done.

7.2.1 Eigenfunction method v.s. Galerkin method

If one consider the PDE problem,

\[ Lu + \rho \ddot{u} = f \]  \hspace{1cm} (7.133)

homogeneous Boundary Conditions  \hspace{1cm} (7.134)

Initial Conditions  \hspace{1cm} (7.135)

suppose to represent the solution and the load in terms of a finite set of \( N \) generalized eigenfunctions \( \phi_n(x) \) (space-discretized solution) of the operator \( L \)

\[ u(x, t) = \sum_{n}^{N} u_n(t)\phi_n(x) \hspace{1cm} f(x, t) = \sum_{n}^{N} f_n(t)\phi_n(x) \]  \hspace{1cm} (7.136)

Then, Equation 7.133 becomes

\[ \sum_{n}^{N} \rho \ddot{u}_n\phi_n + \sum_{n}^{N} u_n L\phi_n = \sum_{n}^{N} f_n\phi_n \rightarrow \sum_{n}^{N} \phi_n(x) (\rho \ddot{u}_n + \lambda_n u_n - f_n) = 0 \]  \hspace{1cm} (7.137)

having considered the definition of generalized eigenfunctions and eigenvalues.

**Eigenfunction method**

As for a generic eigenproblem the eigenfunctions of an operator are independent, Eq. 7.137 yields

\[ \rho \ddot{u}_n + \rho \lambda_n u_n = f_n \hspace{1cm} \forall n \]  \hspace{1cm} (7.138)

Furthermore, in the case that

\[ \langle \rho \phi_n(x), \phi_m(x) \rangle = \delta_{nm} \]  \hspace{1cm} (7.139)

\[ i.e., \text{if and only if the operator } L \text{ is selfadjoint, then} \]

\[ f_n(t) = \langle f(x), \phi_n(x) \rangle \]  \hspace{1cm} (7.140)
• **Galerkin method**

Now, if Eq. 7.133 is generically approximated using a finite set of \( N \) independent functions \( \psi_n(x) \)'s so that

\[
u(x,t) = \sum_{n} v_n(t) \psi_n(x) \quad (7.141)\]

by projecting Eq. 7.133 on each function \( \psi_n(x) \), one has

\[
\sum_{n} \ddot{v}_n \langle \rho \psi_n, \psi_k \rangle + \sum_{n} v_n \langle L \psi_n, \psi_k \rangle = \langle f, \psi_k \rangle \quad k = 1, 2, ..., N \quad (7.142)
\]

or

\[
\sum_{n} M_{kn} \ddot{v}_n + \sum_{n} K_{kn} v_n = f_k \quad k, n = 1, 2, ..., N \quad (7.143)
\]

where

\[
K_{kn} := \langle \psi_n(x), L \psi_k(x) \rangle \quad M_{kn} := \langle \rho \psi_n(x), \psi_k(x) \rangle \quad f_n := \langle f, \psi_n \rangle \quad (7.144)
\]

are stiffness matrix, mass matrix and generalized load given by the Galerkin method.

Note that *if and only if* the \( \psi_n(x) \)'s are chosen to be the eigenfunction \( \phi_n(x) \) (which are orthogonal functions), then Eq. 7.143 (which now is diagonal) becomes identical to Eq. 7.138. Thus, in this case, the two approaches coincide in the sense that they yield the same equations.

Indeed this concept is also related to the concept of *projection* and *component* of a vector in a finite dimensional space: indeed, if one refers to a not-orthogonal basis, the concept of projection and the concept of component typically do not coincide. The *infinite dimension* vector given by the base PDE problem \( \ddot{u} + Lu - f \) is imposed to be equal to zero by the Eq. 7.133: by using the eigenfunction method, on the base of the assumed independence of the eigenfunctions, the *components* of this vector are equivalently set equal to zero obtaining the final Eq. 7.138.

On the other hand, by using the Galerkin method, the *projections* of this vector on the same orthogonal basis are imposed to be zero. Thus, it is apparent that only using an orthonormal basis, i.e., if and only if the operator \( L \) is *self-adjoint*, then the components and the projections on the eigenfunctions coincide. This means that only in this case the equations obtained by the eigenfunction method are exactly the same of those generally obtained by the Galerkin method.

### 7.2.1.1 An example

It is considered as an example of application, the case of a beam supported at the ends and stressed by a uniform load \( p \), Fig.7.2. It has the equation:
Figure 7.2: Application of the Galerkin method: beam supported at the ends with uniform load

\[ EIw^{IV} = p(x) = p \]

with the boundary conditions:

per \( x = 0 \)

- geometric B.C.: \( w = 0 \)
- natural B.C.: \( w'' = 0 \)

per \( x = l^* \)

- geometric B.C.: \( w = 0 \)
- natural B.C.: \( w'' = 0 \)

we look for the solution with the position:

\[ w^*(x) = \sum_k a_k \sin \left( \frac{k\pi x}{l^*} \right) \]

where the basis functions satisfy the boundary conditions of the problem. With the lowest approximation, i.e. considering only one term in the development, it has

\[ w^*(x) = a_1 \sin \left( \frac{\pi x}{l^*} \right) \]

and then the error function becomes:

\[ \varepsilon(x) = EIw^{IV*} - p = \frac{EI\pi^4}{l^*4} - a_1 \sin \frac{\pi x}{l^*} - p \]

and the coefficient \( a_1 \) is determined by imposing the condition of orthogonality:

\[ \varepsilon(x, \phi_1) = \int_0^{l^*} \varepsilon(x) \sin \frac{\pi x}{l^*} dx = 0 \]
from which we obtain:

\[
\int_{0}^{l^*} \left( \frac{EI \pi^4}{l^*^4} a_1 \sin \frac{\pi x}{l^*} - p \right) \sin \frac{\pi x}{l^*} dx = 0
\]

and finally:

\[
a_1 = \frac{p}{EI \pi^4} \int_{0}^{l^*} \sin \frac{\pi x}{l^*} dx = \frac{4}{\pi^5 EI} p l^*^4
\]

then we obtain the arrow in the center of the beam:

\[
w^*(x = l^*/2) = \frac{4}{\pi^5 EI} p l^*^4 = 0.013071 \frac{p l^*^4}{EI}
\]

which results a value very close to that of the exact solution which is given by:

\[
w \left( x = \frac{l^*}{2} \right) = 0.013021 \frac{p l^*^4}{EI}
\]

You can follow an easier route by imposing (as suggested by the collocation method) the cancellation of the error in the point \(x = \frac{l^*}{2}\) and then:

\[
w^*(x) = a_1 \sin \left( \frac{\pi x}{l^*} \right)
\]

\[
\varepsilon(x) = \frac{EI}{l^*^4} \pi^4 a_1 \sin \left( \frac{\pi x}{l^*} \right) - p
\]

\[
\varepsilon(l/2) = \frac{EI}{l^*^4} \pi^4 a_1 - p = 0
\]

whence:

\[
a_1 = \frac{p l^*^4}{EI \pi^4} = 0.01026 \frac{p l^*^4}{EI}
\]

which leads to an underestimation of the effective value of the arrow in the center of the beam:

\[
\bar{w}(x = \frac{l^*}{2}) = 0.01026 \frac{p l^*^4}{EI}
\]

Note how, by choosing the type of solution:

\[
w^*(x) = a_1 \sin \left( \frac{\pi x}{l^*} \right)
\]

can only impose a condition of zero error and, given the symmetry of the problem, the most appropriate is \(x = l^*/2\). Naturally with more \(a_k\) you have more points for error conditions, obtaining a better approximation. Note how, using the same position \(w^*(x) = a_1 \sin \left( \frac{\pi x}{l^*} \right)\), the Galerkin method comes to a better approximation.
7.2.2 Galerkin method v.s. Rayleigh-Ritz method

Let us consider the base problem given by Eq. 7.133 written in the frequency domain with initial condition equal to zero\(^{13}\)

\[
L\tilde{u} - \omega^2 \rho \tilde{u} = \tilde{f} \tag{7.145}
\]

homogeneous Boundary Conditions \(\tag{7.146}\)

where the symbol \(\tilde{\text{\ }}\) represents the Fourier transform of a function of time.

As the Rayleigh-Ritz method is essentially based on a discretization of a minimum functional problem, the first step is to establish the conditions under which the problem given by the Eqs. 7.145 and 7.146 can be rewritten as stationary problem for a functional, \(i.e.,\) if there exists a variational problem corresponding to the previous one.

**Theorem**

The variational problem

\[
\frac{1}{2} \left( \langle \hat{u}, \hat{L}\hat{u} \rangle - \omega^2 \langle \rho \hat{u}, \hat{u} \rangle \right) - \left( \langle \hat{\tilde{u}}, \tilde{f} \rangle \right) = \text{stationary} \quad \hat{u} \tag{7.147}
\]

is equivalent to that given by Eqs. 7.145 and 7.146 if and only if the operator \(L\) is selfadjoint, \(i.e.,\)

\[
\langle \hat{L}\hat{u}, \hat{v} \rangle - \langle \hat{u}, \hat{L}\hat{v} \rangle = 0 \tag{7.148}
\]

The proof of the previous theorem can be performed considering the function \(\hat{u}(x) + \varepsilon \tilde{\eta}(x)\), where \(\tilde{\eta}(x)\) is an arbitrary function satisfying the same boundary condition than \(\hat{u}(x)\) whereas \(\varepsilon\) is a scalar variable. This corresponds to assume that \(\hat{u}(x)\) satisfies to Eq. 7.147 and then one can define an arbitrary function \(\hat{u}(x) + \varepsilon \tilde{\eta}(x)\) using a scalar \(\varepsilon\) and a function \(\tilde{\eta}(x)\) which is also arbitrary but fixed: in this way one can correctly substitute the stationary problem with respect to a function with a stationary (standard) problem with respect to a scalar variable. Thus, the unknown \(\hat{u}\) satisfying Eq. 7.147 can be obtained by\(^{14}\)

\[
\frac{d}{d\varepsilon} \left[ \frac{1}{2} \langle \hat{u} + \varepsilon \tilde{\eta}, L(\hat{u} + \varepsilon \tilde{\eta}) \rangle - \frac{1}{2} \omega^2 \langle \rho(\hat{u} + \varepsilon \tilde{\eta}), \hat{u} + \varepsilon \tilde{\eta} \rangle \right] \bigg|_{\varepsilon=0} = 0 \tag{7.149}
\]

\(^{13}\)Note that Eq. 7.145 for \(\tilde{f} = 0\) represents an eigenproblem whereas for \(\omega^2 = 0\) represents a (static) load problem.

\(^{14}\)Note that setting \(\varepsilon = 0\) in Eq. 7.149 guaranties to obtain the function \(\hat{u}\) that is the solution of Eq. 7.147 by definition.
The above Equation yields
\[
\frac{1}{2} \langle \tilde{\eta}, L \tilde{u} \rangle + \frac{1}{2} \langle \tilde{u}, L \tilde{\eta} \rangle - \frac{1}{2} \omega^2 \langle \rho \tilde{\eta}, \tilde{u} \rangle - \frac{1}{2} \omega^2 \langle \rho \tilde{u}, \tilde{\eta} \rangle - \langle \tilde{\eta}, \tilde{f} \rangle = 0
\] (7.150)
i.e., if \( L \) is self-adjoint
\[
\langle L \tilde{u} - \omega^2 \rho \tilde{u} - \tilde{f}, \tilde{\eta} \rangle = 0
\] (7.151)

Then, as \( \tilde{\eta} \) is assumed to be arbitrary,\(^\text{15}\) the Eq. 7.151 is valid if (and only if) the Eq. 7.145 is valid too.

Now, the Rayleigh-Ritz discretization method consists of solving Eqs. 7.145 and 7.146, in the essential hypothesis of being \( L \) selfadjoint, i.e., by the demonstrated theorem, in solving the alternate Eq. 7.147 after choosing a finite set of \( N \) base of independent functions \( \psi_n(x) \), or
\[
\tilde{u}(x, \omega) = \sum_{n} \tilde{v}_n(\omega) \psi_n(x)
\] (7.152)

Note that the functions \( \psi_n(x) \) satisfy the (homogeneous) boundary conditions whereas the functions \( \tilde{v}_n \) assume the role of state-space variable of the problem. Substituting Eq. 7.152 in Eq. 7.147, one has
\[
\frac{1}{2} \sum_{n} \sum_{m} \langle \psi_n(x), L \psi_m(x) \rangle \tilde{v}_n \tilde{v}_m - \frac{1}{2} \omega^2 \sum_{n} \sum_{m} \langle \rho \psi_n(x), \psi_m(x) \rangle \tilde{v}_n \tilde{v}_m - \sum_{m} \langle \tilde{f}, \psi_m(x) \rangle \tilde{v}_m = 0
\]

Note that the stationary conditions are now with respect to the \( N \) variables \( \tilde{v}_n \) and not with respect to the \( \tilde{u}(x) \). Thus, taking the derivative with respect to \( \tilde{v}_n \) in order to impose the stationary condition, one obtains
\[
\sum_{m} \left( K_{nm} \tilde{v}_m - \omega^2 M_{nm} \tilde{v}_m \right) = \tilde{f}_n \quad i.e. \quad K \tilde{v} - \omega^2 M \tilde{v} = \tilde{f}
\] (7.153)

where the above matrices and generized load (in time domain) are given by Eq. 7.144. Note that the simmetry of the metrices \( K \) and \( M \) is based on the simmetry (or selfadjointness) of the structural operator \( L \).

Now, a comparison between the Rayleigh-Ritz method and the Galerkin method (see Section 7.2.1) will be done.\(^\text{16}\) The Galerkin method applied on the problem given by Eq. 7.145 and

\(^{15}\)This statement is also known as lemma of calculus of variations.

\(^{16}\)Indeed, the functions \( \psi_n(x) \) used for the Galerkin method are to satisfy more homogeneous boundary conditions - i.e., the so called geometric and natural - with respect to those used for the Rayleigh-Ritz method (only the geometric homogeneous boundary conditions): this is due to the fact that the variational equation associated to the Rayleigh-Ritz method automatically satisfy the natural boundary conditions.
\[ L\ddot{u} - \omega^2 \ddot{u} = \ddot{f} \quad \iff \quad \langle L\ddot{u} - \omega^2 \ddot{u} - 2\ddot{f}, \ddot{u} \rangle = \text{staz.} \]

Table 7.1: Comparison between Galerkin method and Rayleigh-Ritz method

7.146 and assuming for the solution the spectral decomposition given by Eq.7.141 or Eq. 7.152 in frequency domain, will give, after the projection on the function \( \psi_n(x) \), the Eq. 7.143 or, in frequency domain,

\[
\sum_{m} \langle L\ddot{\psi}_n(x), \ddot{\psi}_m(x) \rangle \ddot{v}_m - \omega^2 \sum_{m} \langle \rho \ddot{\psi}_n(x), \ddot{\psi}_m(x) \rangle \ddot{v}_m - \langle \ddot{f}(x), \ddot{\psi}_n(x) \rangle = 0 \quad n = 1, 2, ..., N(7.154)
\]

which is identical to Eq. 7.153. Indeed, this equivalence for the discretized systems (see Tab. 7.1) is strictly connected with the hypothesis that there exists a variational principle (Eq. 7.147) alternative to the differential problem (Eq. 7.145) and this is true if and only if \( L \) is selfadjoint; in other words, the Galerkin method is optimal if (and only if) \( L \) is selfadjoint. It is also worth to point out that the Galerkin method can be always applied whereas the Rayleigh-Ritz method only if there exists a variational principle.

A final comment can be done in the case that the functions \( \psi_n(x) \) are the eigenfunctions of the operator \( L \), i.e., \( \psi_n(x) \equiv \phi_n(x) \) (and \( \ddot{v}_n \) are state-space variables instead of \( \ddot{\psi}_n \)): in this case, in the further hypothesis the operator \( L \) is selfadjoint, not only the equations obtained with the Rayleigh-Ritz method (Eq. 7.153) are the same of those of Galerkin method (as shown before), but these equations coincides also with those obtained by the eigenfunctions method (see Eq. 7.138 in the frequency domain). Note also that in this case the mass and the stiffness matrix are diagonal.

### 7.2.2.1 Examples

- As an example of application of the [Ritz method] a beam fixed at one end and solicited with a concentrated force denoted by \( P \) at the free end is considered, Fig. 7.3 one seek the solution with a polynomial growth:

\[
w^*(x) = \sum_k a_k \Phi_k(x) = \sum_k a_k x^k
\]

(7.155)
Figure 7.3: Application of Ritz method: cantilevered beam with a tip pin force.

This solution must satisfy the boundary conditions, only the geometric ones because it is a variational approach:

\[ x = 0 \quad w^* = 0 \quad w'^* = 0 \]

these conditions, rewritten in terms of the unknown coefficients, become:

\[ a_0 = 0 \quad a_1 = 0 \]

and then developing for the function \( w^*(x) \) becomes:

\[ w^*(x) = \sum_k a_k x^k \quad k = 2, \ldots, N \]

the energy is now calculated as a function of the unknowns \( a_k \) and one has:

\[ V(w) = \int_0^l \frac{EI}{2} w''^2 dx - Pw_1^* \]

if the minimum development is considered, with only one term one has:

\[ w^*(x) = a_2 x^2 \]

then:

\[ \Pi(a_2) = \int_0^l \frac{EI}{2} 4a_2^2 dx - P a_2 l^2 = 2E I a_2^2 l - P a_2 l^2 \]

and from the condition of extremalization:

\[ \frac{\partial \Pi}{\partial a_2} = 0 \]
the determination of the unknown \( a_2 \) is obtained:

\[
a_2 = \frac{Pl^*}{4EI}
\]

and then the solution is obtained:

\[
w^*(x) = \frac{Pl^*}{4EI} x^2
\]

at the free end, \( x = l^* \), one has:

\[
w^*(l^*) = w^*_L = \frac{Pl^*}{4EI}
\]

value that is significantly different from the exact solution of the problem given by:

\[
w^*_L = \frac{Pl^*}{2EI}
\]

if instead a development with two terms is considered:

\[
w^*(x) = a_2 x^2 + a_3 x^3
\]

one has:

\[
\Pi(a_2, a_3) = \frac{EI}{2} \int_0^l (2a_2 + 6a_3 x)(2a_2 + 6a_3 x) dx - P (a_2 l^2 + a_3 l^3)
\]

from the conditions:

\[
\frac{\partial \Pi}{\partial a_2} = 0 \\
\frac{\partial \Pi}{\partial a_3} = 0
\]

one obtains:

\[
a_2 = \frac{Pl^*}{2EI} \quad a_3 = -\frac{P}{6EI}
\]

ie the solution:

\[
w^*(x) = \frac{P}{2EI} \left( l^* x^2 - \frac{x^3}{3} \right)
\]

the 7.156 calculated in the extreme free, \( x = l^* \), becomes:

\[
w^*(l^*) = w^*_L = \frac{Pl^3}{3EI}
\]

which is just the exact value of the solution, and this is due to the fact that the solution is a linear combination of basic functions that were selected with the 7.155 in search of the
solution. It is observed that the selected approximation type contains the quadratic term and this allows to represent the state of constant curvature:

\[ w'' = a_2 = \text{const.} \]

If this term had been ruled out with an approximation for the displacement of the type:

\[ w^* (x) = a_3 x^3 + a_4 x^4 \]

would have obtained a convergence towards an incorrect value for the energy.

- Let us now consider the example of the supported beam, stressed with a uniform load \( p \) that has been previously solved by the Galerkin method Fig. 7.2. In this case the potential energy to be minimized is given by:

\[ \Pi (w) = \int_0^l \frac{EI}{2} w''^2 dx - \int_0^l p w dx \]

the boundary conditions to be observed are only those geometric or essential, i.e.

\[ x = 0 \quad w = 0 \]
\[ x = l^* \quad w = 0 \]

For the function of moving a series expansion is considered in terms of sine functions of the type:

\[ w^* (x) = \sum_k a_k \sin \frac{k\pi x}{l} \]

in order to meet the boundary conditions of the problem an approximation with a single term is considered:

\[ w^*(x) = a_1 \sin \frac{\pi x}{l} \]

for the energy one has:

\[ \Pi (w^*) = \int_0^l \frac{EI}{2} \left( \frac{\pi}{l} \right)^4 a_1^2 \sin \frac{\pi x}{l} \sin \frac{\pi x}{l} dx - \int_0^l p a_1 \sin \frac{\pi x}{l} dx \]

in terms of the unknown \( a_1 \) one has:

\[ \Pi (a_1) = a_1^2 \left( \frac{\pi}{l} \right)^4 \frac{EI}{2} \int_0^l \sin^2 \frac{\pi x}{l} dx - a_1 p \int_0^l \sin \frac{\pi x}{l} dx \]

from the condition of minimum:

\[ \frac{\partial \Pi}{\partial a_1} = 0 \]
is obtained:

\[ a_1 \left(\frac{\pi}{l}\right)^4 EI \int_0^l \sin^2 \frac{\pi x}{l} \, dx - p \int_0^l \sin \frac{\pi x}{l} \, dx = 0 \]

and then:

\[ a_1 = \frac{p \int_0^l \sin \frac{\pi x}{l} \, dx}{EI \left(\frac{\pi}{l}\right)^4 \int_0^l \sin^2 \frac{\pi x}{l} \, dx} = \frac{4 \, p l^4}{\pi^5 EI} = 0.13071 \frac{pl^4}{EI} \]

that provides the same solution which had been determined previously as Galerkin. It is noted that the functional basis of choice for solving the problem with the integral approach must respect only the essential boundary conditions while in the Galerkin method must satisfy both the essential conditions and the natural ones. This means that the choice of the functional basis is more simple in the Ritz approach, since it has to respond to a smaller number of conditions, and this is an advantage of this method, but it also means that the functional basis is less tied to the problem that have to be solved, and then the convergence may be slower than the Galerkin approach.

- In the case of the simply supported beam with a distributed constant load \( p(x) = p \), using a Ritz approach with reference to a polynomial development like

\[ w^*(x) = a_0 + a_1 x + a_2 x^2 \]

only the two geometric boundary conditions are imposed:

\[ w(0) = 0 \Rightarrow a_0 = 0 \]
\[ w(l) = 0 \]

then

\[ w^*(x) = a_1 x + a_2 x^2 \]

the second condition

\[ w(l) = a_1 l + a_2 l^2 = 0 \]

leads to:

\[ a_1 = -a_2 l \]

then

\[ w^*(x) = a_2 \left( x^2 - l x \right) \]
whence

\[ w''(x) = 2a_2 \text{ (costante)} \]

in this approach the natural boundary conditions, in this case relating to the moments, have not to be imposed. One also has:

\[ \Pi(w) = \int_0^l \frac{EI}{2} \left( \frac{d^2w}{dx^2} \right)^2 dx - \int_0^l pwdx \]

i.e.

\[ \Pi(w) = \int_0^l \frac{EI}{2} (2a_2)^2 dx - \int_0^l p a_2 \left( x^2 - lx \right) dx = \frac{EI}{2} 4a_2^2 l - p a_2 \left( \frac{l^3}{3} - \frac{l^3}{2} \right) \]

and then:

\[ \Pi(w) = 2a_2^2 EIl + pa_2 \frac{l^3}{6} \]

by the condition:

\[ \frac{\partial \Pi}{\partial a_2} = 0 \]

one obtains:

\[ a_2 = -\frac{l^3}{24EI} = -\frac{l^2}{24EI} \]

and then:

\[ w^*(x) = -\frac{l^2}{24EI} \left( x^2 - lx \right) \]

the \( w^*(x) \) calculated in the point \( x = \frac{l}{2} \) is:

\[ w^* \left( \frac{l}{2} \right) = -\frac{l^2}{24EI} \left( \frac{l^2}{4} - \frac{l^2}{2} \right) = \frac{1}{96} \frac{pl^4}{EI} = 0.0104 \frac{pl^4}{EI} \]

The obtained value is far from the exact solution \( w \left( \frac{l}{2} \right) = 0.01302 \frac{pl^4}{EI} \). Still in the case of the simply supported beam with distributed constant load, we look for the solution as Galerkin with the position:

\[ w^*(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \]

imposing the conditions:

\[ w^*(0) = 0 \]
\[ w^{*''}(0) = 0 \]
that lead to: \( a_0 = 0 \) e \( a_2 = 0 \) and the conditions

\[
\begin{align*}
    w^*(l) &= 0 \\
    w''*(l) &= 0
\end{align*}
\]

one finds:

\[
\begin{align*}
    a_1 &= -a_3 l^2 - a_4 l^3 \\
    a_3 &= -2a_4 l
\end{align*}
\]

then

\[
a_1 = a_4 l^3
\]

and the proposed solution becomes:

\[
w^*(x) = a_4 l^3 x - 2a_4 l x^3 + a_4 x^4 = a_4 \left( l^3 x - 2l x^3 + x^4 \right)
\]

The error function is:

\[
\epsilon(x) = EIw^{IV} - p = 24EIa_4 - p
\]

the unknown coefficient \( a_4 \) is determined from the condition of Galerkin:

\[
\epsilon(x, \Phi_1) = \int_0^l (24EIa_4 - p)(l^3 x - 2l x^3 + x^4) \, dx
\]

whence

\[
a_4 = \frac{p \int_0^l (l^3 x - 2l x^3 + x^4) \, dx}{24EI \int_0^l (l^3 x - 2l x^3 + x^4) \, dx}
\]

then

\[
a_4 = \frac{p}{24EI}
\]

whence:

\[
w^*(x) = \frac{p}{24EI} \left( l^3 x - 2l x^3 + x^4 \right)
\]

Then the arrow in the center of the beam is:

\[
w^* \left( \frac{l}{2} \right) = \frac{p}{24EI} \left( \frac{l^4}{2} - \frac{l^4}{4} + \frac{l^4}{16} \right) = 0.13021 \frac{pl^4}{EI}
\]

value that corresponds to the exact solution of the problem.
7.2.3 Some issues on approximated-Eigenfunctions

The issues presented in this Section for the eigenfunction method and approximate eigenfunction method are purely identical to those shown in Section 6.6.1 for modes of vibration and approximate modes of vibration for the case of general vibrational problem. Specifically, we shall follow two apparently different procedure: i) assume as base for the eigenfunction method (or Galerkin method, because we assume that \( L \) is selfadjoint) approximate eigenfunction \( \tilde{\phi}_n(x) \) (see later for definitions); ii) assume as base for the Galerkin method arbitrary function \( \hat{\phi}_n(x) \) and then diagonalize the obtained N-dimensional second order ordinary differential equation problem (see Fig. 7.4).

- Let us consider the problem given by the Eqs. 7.133, 7.134, and 7.135; then, let us assume as base for the (approximate) solution \( M \) approximated eigenfunctions \( \tilde{\phi}_n(x) \) such as

\[
 u(x, t) = \sum_{n}^{M} \tilde{u}_n(t) \tilde{\phi}_n(x) \tag{7.157}
\]

\(^{17}\)For example, the function of the Finite-Element method which have the advantage to imply state-space variable \( u_n \) having the meaning of value of the unknown function \( u(x) \) in a node of the structure.
The meaning of “approximated eigenfunctions” will be clarified: these functions will satisfy the eigenproblem:

\[ L \bar{\phi}_n(x) = \bar{\lambda}_n \bar{\phi}_n(x) \quad \text{Homogeneous B.C.} \quad n = 1, 2, \ldots, M \] (7.158)

(\(\bar{\lambda}_n\) are the unknown approximated eigenvalues) with the essential issues that they are searched in a finite-dimensional space, i.e., \(n = 1, 2, \ldots, M\).

Then, in order to clarify the role of the approximated eigenvalues and eigenfunctions let us apply the Galerkin method to Eq. 7.133 considering a finite number \(N\) of known functions \(\hat{\phi}_n(x)\) such as

\[ u(x, t) = \sum_{n}^{N} \ddot{u}_n(t) \hat{\phi}_n(x) \] (7.159)

Then, one obtains

\[ \sum_{n}^{N} \left\langle \hat{\phi}_n(x), \hat{\phi}_k(x) \right\rangle \dddot{u}_n + \sum_{n}^{N} \left\langle L\hat{\phi}_n(x), \hat{\phi}_k(x) \right\rangle \dot{u}_n = \left\langle f, \hat{\phi}_k(x) \right\rangle \quad k = 1, \ldots, N \] (7.160)

or

\[ \hat{M}\dddot{u} + \hat{K}\dot{u} = \hat{f} \] (7.161)

with \(\hat{M}_{nk} := \left\langle \hat{\phi}_n(x), \hat{\phi}_k(x) \right\rangle\), \(\hat{K}_{nk} := \left\langle L\hat{\phi}_n(x), \hat{\phi}_k(x) \right\rangle\), and \(\hat{f}_k := \left\langle f, \hat{\phi}_k(x) \right\rangle\). Now, a standard eigenproblem can be associated to the free vibration problem associated Eq. 7.161 obtaining

\[ \hat{K}\hat{z}^{(m)} - \bar{\lambda}_m \hat{M}\hat{z}^{(m)} = 0 \] (7.162)

where \(\bar{\lambda}_m\) and \(\hat{z}^{(m)}\) are the \(M\) eigenvalues and eigenvectors associated to the matrices \(\hat{M}\) and \(\hat{K}\). Because of the symmetry and positiveness of the matrices \(\hat{M}\) and \(\hat{K}\), one obtains real and positive eigenvalues \(\bar{\lambda}_m\) and orthogonal eigenvectors \(\hat{z}^{(m)}\), i.e.,

\[ \hat{z}^{(p)T}\hat{K}\hat{z}^{(q)} = \delta_{pq}\bar{\lambda}_p \quad \hat{z}^{(p)T}\hat{M}\hat{z}^{(q)} = \delta_{pq} \] (7.163)

assuming to normalize the eigenvectors in order to have generalized masses equal to 1.

Now, if one defines the approximated eigenfunctions \(\bar{\phi}_m(x)\) as given by using \(N\) known functions \(\hat{\phi}_n(x)\) as base-functions such as

\[ \bar{\phi}_m(x) := \sum_{n}^{N} \hat{z}^{(m)T}_n \hat{\phi}_n(x) \] (7.164)

18In the standard linear computational structural dynamics, \(\hat{\phi}_n(x)\)'s represent the base functions of the Finite-Element method: consequently, the state-space variable \(\ddot{u}_n\) represent the physical displacement associated to the \(n\)th degree of freedom.
and the *approximated eigenvalues* $\bar{\lambda}$ as those given by Eq. 7.162, one can demonstrate that they satisfy Eq. 7.158; indeed, they satisfy the first $M$ projections of such equation on the $M$ approximated eigenfunctions $\bar{\phi}_r(x)$ (i.e., $\langle L\bar{\phi}_s - \bar{\lambda}_s \bar{\phi}_s, \bar{\phi}_r \rangle = 0$, $r = 1, ..., M$): this can be shown considering the Eq. 7.163 and 7.164 and evaluating the inner products

\[
\langle \bar{\phi}_r(x), \bar{\phi}_s(x) \rangle = \sum_{n \neq m \neq s} \hat{M}_{nm} z_n^{(r)} z_m^{(s)} = z_n^{(r)} z_m^{(s)} \delta_{rs} = \delta_{rs} \bar{\lambda}_r
\]

(7.165)

\[
\langle \bar{\phi}_r(x), L\bar{\phi}_s(x) \rangle = \sum_{n \neq m \neq s} \hat{K}_{nm} z_n^{(r)} z_m^{(s)} = z_n^{(r)} K z_m^{(s)} = \delta_{rs} \bar{\lambda}_r
\]

(7.166)

demonstrating that these $M$ approximated eigenvalues and eigenvector identically satisfy the first $M$ projections of Eq. 7.158 on the function $\bar{\phi}_n(x)$: in other words, although these approximated functions are orthogonal (as the discrete eigenvector $z^{(s)}$ are orthogonal), they are not a complete base of functions because they cannot span all the functional space which has infinitive dimension.

Once the orthogonality of the *approximated eigenfunctions* has been shown, one can obtain the discretized (diagonalized) forced problem given by

\[
\ddot{u}_p + \bar{\lambda}_p \dot{u}_p = \bar{f}_p \quad p = 1, 2, ..., M
\]

(7.167)

with

\[
\bar{f}_p := \langle f, \phi_p(x) \rangle = \sum_n \langle f, \hat{\phi}_n(x) \rangle \hat{z}_n^{(p)} = z^{(p)} \hat{f}
\]

(7.168)

*• Next, we shall show an equivalent way to obtain the final Eq. 7.167. Let us consider again the problem given by the Eqs. 7.133, 7.134, and 7.135; then, let us assume as base for the (approximate) solution $N$ arbitrary functions $\hat{\phi}_n(x)$ such as

\[
u(x, t) = \sum_{n} \hat{u}_n(t) \hat{\phi}_n(x)
\]

(7.169)

Applying the Galerkin method to the Eqs. 7.133, 7.134, and 7.135, and considering Eq. 7.169, one has

\[
\sum_{n} \ddot{\hat{u}}_n \langle \hat{\phi}_n(x), \hat{\phi}_k(x) \rangle + \sum_{n} \hat{u}_n \bigl\langle L \hat{\phi}_n(x), \hat{\phi}_k(x) \bigr\rangle = \langle f(x, t), \hat{\phi}_k(x) \rangle \quad k = 1, ..., N(7.170)
\]
or

\[
\ddot{\hat{M}}\hat{u} + \hat{K}\hat{u} = \hat{f}
\] (7.171)

considering the usual meaning for the introduced symbols. Now, if one introduces the (linear) coordinate transformation for the state-space variable \( \hat{u} \) as in the following

\[
\hat{u} = z^{(1)}y_1 + z^{(2)}y_2 + \ldots + z^{(M)}y_M = Zy
\] (7.172)

where \( Z \) is the \( N \times M \) matrix of the eigenvectors as obtained by the eigenproblem given by Eq. 7.162. Then, pre-multiplying Eq. 7.171 by \( Z^T \) and considering the Eq. 7.163 one has

\[
\ddot{y} + \bar{\Lambda}y = Z^T\hat{f}
\] (7.173)

where \( \bar{\Lambda} \) is the diagonal matrix of the approximated eigenvalues \( \bar{\lambda}_m \).

Note that this Equation is identical to Eq. 7.167 which demonstrates that \( \bar{u} \equiv y \), i.e., that the solving equation of the approximated eigenfunction method can also be directly obtained diagonalizing the equation given by the Galerkin method via the eigenvectors associated to the corresponding eigenproblem. Indeed, this has been essentially obtained through the linear change of state-space co-ordinate given by Eq. 7.172; note that this has been equivalently obtained, as shown in the previous item, using the linear change of base functions of the Galerkin method given by the Eq. 7.164 (see Tab. 7.2).

### 7.2.4 Fredholm Alternative in the Eigenfunctions/Eigenvectors Method

In this Section the concept of Fredholm alternative will be parallely introduced in three different applications: \( i \) a PDE with the eigenfunction method (in frequency domain; \( ii \) an ODE problem solved with the eigenvector method (in frequency domain; \( iii \) an algebraic linear problem.
1. Consider the Frequency Response Function *continuum* problem given by Eq. 7.145 with exact solution given by Eq. 7.96: the resonance condition encountered when the load has a harmonic component close to an angular natural frequency $\sqrt{\lambda_n}$ may also not occurs if

$$\hat{f}_n = \langle \hat{f}, \phi_n \rangle \equiv 0,$$

*i.e.*, if the load is orthogonal to the corresponding eigenfunction. Then, in this case the solution may not be *undetermined*.

2. Now, consider the (linear) discretized problem given by the Lagrange equation in frequency domain

$$\hat{K}\ddot{q} - \omega^2 \hat{M}\ddot{q} = \hat{e}$$

(7.174)

Expressing the solution by the eigenvectors method, *i.e.*, changing the co-ordinate $\ddot{q} = Z\ddot{\tilde{q}}$, one obtains (see Eq. 6.34 and also 6.85 for second order systems)

$$\ddot{\tilde{q}} = Z \begin{bmatrix}
\ddots \\
1
\frac{1}{\lambda_n - \omega^2}
\ddots \\
\ddots
\end{bmatrix} Z^T \ddot{\tilde{e}}$$

(7.175)

Then, one has the same comment done before but now for finite-dimensioned dynamical system: the resonance condition encountered when the load has a harmonic component close to an angular natural frequency $\sqrt{\lambda_n}$ may also not occurs if $\ddot{\tilde{e}}_n = z^{(n)}\ddot{\tilde{e}} \equiv 0$, *i.e.*, if the load is orthogonal to the corresponding eigenvector. Then, in this case the solution may not be *undetermined*.

3. Finally, let us consider the linear algebraic system (for example, the Eq. 7.175 written, for a prescribed $\omega$, as $H^{-1}(\omega)\ddot{q} = \ddot{\tilde{e}}$),

$$Ax = b$$

(7.176)

with $A$ square symmetric matrix (in analogy with the previous use of symmetric operator and matrix). As well known, if $detA \neq 0$ (namely, this is the case of the resonance condition), one has a unique solution both for the homogeneous and for the non homogeneous problem: this solution is $x = A^{-1}b$.

Conversely, if one has $detA = 0$, then one eigenvalue of $A$ is equal to zero\(^{19}\) and that implies

$$A z = 0 \quad \text{or} \quad z^T A = 0$$

(7.177)

---

\(^{19}\)The determinant is the matrix invariant equal to the product of all the eigenvalues.
where $z$ is the eigenvector associated to the eigenvalue equal to zero. Then, in this case, the only possibility (so-called Fredholm alternative) that the problem given by Eq. 7.176 has a solution is that (see Eq. 7.177)

\[(z^T A) x = z^T b = 0\] (7.178)

i.e., that the vector of the known terms is orthogonal to the eigenvector $z$ that is the non trivial solution of the homogeneous problem. Note also that the condition $\text{det}(A) = 0$ not only is equivalent to $z^T A = 0$, but this corresponds also to say that the rows of the matrix $A$ are linearly dependent and the entries of $z$ represents the coefficients of such linear combination.

### 7.3 Linear operators in three-dimensional linear elasticity

In this section some issues presented for scalar linear operator, will be extended to the three-dimensional structural operator. The objective is to introduce the essential theoretical tool usefull to set up the boundary element method for elastostatic problem. Moreover, the extension of the concept of structural operator for the 3D problem will also allow to re-interprete the Lagrangian equation of motion (Cap. 4) as a equation of eigenfunction method by using truncated eigenfunctions.

#### 7.3.1 Linear dynamic equations for elasticity

We assume that the relationship between the stress and the strain tensor is linear:

\[\tau_{ij} = C_{ijkm} \varepsilon_{km}\] (7.180)

with (linear strain-tensor expression in Cartesian coordinates)

\[\varepsilon_{km} = \frac{1}{2}(u_{k,m} + u_{m,k})\] (7.181)

The Cauchy law for dynamic equilibrium is given by

\[\rho \ddot{u}_i = \tau_{ij,j} + \rho f_i\] (7.182)

\[\text{Note that the Hookean case considered in the Section 3.3.1 is a special case of this one; indeed, if the solid material is isotropic, the Hook law given by Eq. 3.67 can be written in term of the constants } C_{jikm} \text{ as}\]

\[C_{jikm} := \lambda \delta_{ij} \delta_{km} + \mu (\delta_{ik} \delta_{jm} + \delta_{im} \delta_{kj})\] (7.179)

where $\lambda$ and $\mu$ are the Lame constants.
Combining with the above expressions yields

\[ \varrho \dddot{u}_i = \left( C_{ijkm} \frac{u_{k,m} + u_{m,k}}{2} \right)_j + \varrho f_i \]  

(7.183)

Applying on the consequence of two symmetry properties of \( C_{ijkm} \), the Equation 7.183 becomes

\[ \varrho \dddot{u}_i = (C_{ijkm} u_{k,m})_j + \varrho f_i \]  

(7.184)

This may be written as

\[ \varrho \dddot{u}_i + L_{ik} u_k = \varrho f_i \]  

(7.185)

where

\[ L_{ik} = -\frac{\partial}{\partial x_j} \left( C_{ijkm} \frac{\partial}{\partial x_m} \right) \]  

(7.186)

The boundary conditions are either

\[ u_k = \text{known} \quad \text{(geometric B.C.)} \]  

(7.187)

or

\[ t_i = \tau_{ij} n_j = C_{ijkm} u_{k,m} n_j = \text{known} \quad \text{(natural B.C.)} \]  

(7.188)

### 7.3.2 Adjoint and selfadjoint operators

We recall the definition of adjoint scalar operator. Given a differential operator \( L \), with boundary-condition operator \( L_{BC} \) we say that the complete operator (i.e., \( L \) plus \( L_{BC} \)) has an adjoint complete operator, \( L^* \) plus \( L_{BC}^* \), if

\[ \langle v, Lu \rangle = \langle L^* v, u \rangle \]  

(7.189)

when \( L_{BC} u = L_{BC}^* v = 0 \).

The complete scalar operator is selfadjoint if and only if \( L = L^* \) and \( L_{BC} = L_{BC}^* \).

For matrix operators we have, with suitable boundary conditions,

\[ \sum_{ij} \langle v_i, L_{ij} u_j \rangle = \sum_{ij} \langle L_{ij}^* v_i, u_j \rangle \]  

(7.190)

i.e., the adjoint of \( L = [L_{ij}] \) is \( L^* = [L_{ji}] \). The complete matrix operator is selfadjoint if and only if \( L = L^* \) and \( L_{BC} = L_{BC}^* \).
7.3.3 Selfadjointness of structural operators

The structural operator $L_{ij}$ defined above is selfadjoint. Indeed (avoiding to represents sums for repeated indeces)

$$\langle v_i, L_{ik}u_k \rangle = -\iiint_V v_i(C_{ijkm}u_{k,m})_j dV$$

$$= \iiint_V v_{ij}C_{ijkm}u_{k,m} dV - \iint_S v_iC_{ijkm}u_{k,m}n_j dS$$

$$= \iiint_V v_{ij}C_{ijkm}u_{k,m} dV - \iint_S v_i t_i(u_k) dS$$

where the Gauss theorem has been applied and where $t_i(u_k)$ means “the $t_i$ due to $u_k$”.

Interchanging $u$ and $v$, $i$ and $k$, subtracting, and using $C_{ijkm} = C_{kmij}$, i.e., linear elastic solid hypothesis (see Sec. 3.2.2) we have

$$\langle v_i, L_{ik}u_k \rangle - \langle u_k, L_{ki}v_i \rangle = -\iint_S [v_i t_i(u_k) - u_k t_k(v_i)] dS$$

or

$$\langle v_i, L_{ik}u_k \rangle = \langle u_k, L_{ki}v_i \rangle$$

(i.e., $L$ = selfadjoint) because of homogeneous B.C. on either $u_k$ or $t_k$. Note that Eq. 7.194 is the statement of the Betti-Castigliano theorem in 3-D linear elasticity (see also Section 7.1.4): it is based on the local property of linear elastic solid given by Eq. 3.60.

7.3.4 Green’s function (or influence function) for static problems*

Let $G(x, x^*)^{(p)}_i$ denote the displacement in $x$ in direction $i$, due to force loaded in $x^*$ in direction $p$, i.e.,

$$L_{ki}G(x, x^*)^{(p)}_i = \delta_{kp}\delta(x - x^*)$$

with boundary condition

$$G_i^{(p)} = 0 \quad \text{on} \ S_1$$

$$t_i^{(p)} = C_{ijkm}G^{(p)}_{k,m}n_j = 0 \quad \text{on} \ S_2$$

$G(x, x^*)^{(p)}_i$ is called the Green function (or influence function) of the operator $L_{ij}$. 
7.3.4.1 Symmetry of $G(x, x^*)^{(p)}_i$

Next, we want to prove that $G(x, x)^{(p)}_i$ is symmetric, in the sense

$$G(x_1, x_2)^{(p)}_i = G(x_2, x_1)^{(i)}_p$$

which means

*The i-component of the displacement in $x_1$, due to a force in direction p in $x_2$ is equal to the p-component of the displacement in $x_2$ due to a force in direction i in $x_1$.*

In order to prove this, note that

$$L_{ik} G^{(p)}_k(x, x_1) = \delta_{ip} \delta(x - x_1)$$

and

$$L_{ki} G^{(q)}_i(x, x_2) = \delta_{kq} \delta(x - x_2)$$

Also

$$\langle G^{(q)}_i(x, x_2), L_{ik} G^{(p)}_k \rangle = \langle G^{(p)}_k, L_{ki} G^{(q)}_i \rangle$$

Hence,

$$\langle G^{(q)}_i(x, x_2), \delta_{ip} \delta(x - x_1) \rangle = \langle G^{(p)}_k(x, x_1), \delta_{kq} \delta(x - x_2) \rangle$$

i.e.,

$$G^{(q)}_p(x_1, x_2) = G^{(p)}_q(x_2, x_1)$$

Note that the function $G^{(q)}_p(x_1, x_2)$ has to satisfy homogeneous boundary conditions.

7.3.5 Green’s integral representation using free-space Green influence function: BEM in elastostatics

Note that the Eq. 7.194 is true in general, i.e., it is true also if the boundary conditions for the adjoint solution $v$ are unknown, i.e., that are not homogeneous: specifically, one could introduce a solution $\tilde{G}_i^{(p)}$ for the (self-)adjoint structural problem satisfying (see Ref. [1])

$$L_{ki} \tilde{G}_i^{(p)}(x, x^*) = \delta_{kp} \delta(x - x^*)$$

(7.205)
but imposing no boundary condition: this function is the free-space Green influence function and it correspond to the physic problem of an influence function for an infinitive linear elastic solid. The only advantage in defining it is that it is can sometimes be evaluated analytically \(^{21}\) (and independently by the boundary condition). Then, setting \(v_{i}^{(p)} := \bar{G}_{i}^{(p)}\) and \(t_{k}^{(p)} = C_{kijm}G_{i,m}^{(p)}n_{j}\) in Eq. 7.194 we have

\[
\langle \bar{G}_{i}^{(p)}, L_{ik}u_{k} \rangle - \langle u_{k}, L_{ki}G_{i}^{(p)} \rangle = -\iint_{S} \left[ \bar{G}_{i}^{(p)} t_{i} - u_{k} t_{k}^{(p)} \right] dS \tag{7.208}
\]

Next note that \(S = S_{1} \cup S_{2}\) and use \(BC\) on \(u_{k}\) and \(t_{i}\). This yields

\[
up(x_{*}) = \iiint_{V} g_{f_{i}} \bar{G}_{i}^{(p)} dV - \iiint_{S} t_{i}^{(p)} u_{i} dS + \iiint_{S} \bar{G}_{i}^{(p)} t_{i} dS \tag{7.209}
\]

where, because of the non homogeneous boundary condition of the actual problem, part of the surface terms are known and part are unknown.

The above equation, when discretized dividing the surface \(S\) in a finite number of panels, and when the contribution of the volume forces is not present, is the base formula for the linear Boundary Element Method (\(BEM\)) in elastostatics.

\(^{21}\)For the three-dimensional problem this solution is analytically given as (Ref. [1])

\[
\bar{G}_{i}^{(p)} = \frac{1}{16\pi(1-\nu)\mu r} [(3 - 4\nu)\delta_{ip} + r_{i} r_{p}] \tag{7.206}
\]

whereas for two-dimensional problem it is given by

\[
\bar{G}_{i}^{(p)} = \frac{1}{8\pi\mu(1-\nu)} \left[ (3 - 4\nu) \ln \frac{1}{r} \delta_{ip} + r_{i} r_{p} \right] \tag{7.207}
\]

where \( \bullet_{,i} := \partial \bullet / \partial x_{j} \), \( \mu \) is a Lame constant, and \( \nu \) is the Poisson modulus.